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On the tensionless limit of gauged WZW models

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Abstract: The tensionless limit of gauged WZW models arises when the level of the underlying Kac-Moody algebra assumes its critical value, equal to the dual Coxeter number, in which case the central charge of the Virasoro algebra becomes infinite. We examine this limit from the world-sheet and target space viewpoint and show that gravity decouples naturally from the spectrum. Using the two-dimensional black-hole coset $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ as illustrative example, we find for $k = 2$ that the world-sheet symmetry is described by a truncated version of W_∞ generated by chiral fields with integer spin $s \geq 3$, whereas the Virasoro algebra becomes abelian and it can be consistently factored out. The geometry of target space looks like an infinitely curved hyperboloid, which invalidates the effective field theory description and conformal invariance can no longer be used to yield reliable space-time interpretation. We also compare our results with the null gauging of WZW models, which correspond to infinite boost in target space and they describe the Liouville mode that decouples in the tensionless limit. A formal BRST analysis of the world-sheet symmetry suggests that the central charge of all higher spin generators should be fixed to a critical value, which is not seen by the contracted Virasoro symmetry. Generalizations to higher dimensional coset models are also briefly discussed in the tensionless limit, where similar observations are made.

Keywords: Conformal Field Models in String Theory, Conformal and W Symmetry, Sigma Models, BRST Quantization.

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1. Introduction

The tensionless limit of string theory is a very fascinating but largely unexplored subject that was first introduced classically in flat space by letting all points of the string move at the speed of light, thus leading to the notion of null strings, [1]. Aspects of their quantization were subsequently studied, [2, 3], and it was also found that the concept of critical dimension does not arise in this case, as the Virasoro algebra contracts to an abelian structure and the corresponding BRST operator squares to zero, [4, 5] for all dimensions of space-time; see also [6] for a more recent discussion of the subject. It is a first indication that the notion of space-time undergoes a drastic modification as on passes from tensile to tensionless strings. This limit also arises naturally in various attempts to formulate a

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sensible expansion of string propagation in highly curved backgrounds, since strings appear to behave as tensionless classical objects in the vicinity of space-time singularities, [7]. Other classical tensionless string models were introduced recently in terms of geometric actions that are alternative to Polyakov's action, and their quantization was investigated in connection with higher spin fields, [8]. On the other hand, there is another interesting approach to the tensionless limit of string theory, which arises directly at the quantum level, and it was brought to light by studying the high energy behavior of string scattering amplitudes at the Planck scale, [9, 10]. Although it is not known whether all these theories are equivalent to each other, or whether they represent different corners of a more general (yet unknown) framework, there is a common element that makes tensionless strings special, namely that the Planck mass becomes zero and all states turn massless as $\alpha' \to \infty$.

It has been suggested that the tensionless limit represents the unbroken phase of string theory, where all states appear on equal footing and they give rise to a huge symmetry group, which subsequently breaks and gives masses to the string states at lower energy scales, [11]; see also [12] for a more recent discussion in terms of higher spin symmetries. As such, it could be used to reveal the fundamental symmetry principles of strings at the Planck scale, and there are also indications that the theory might be topological in vein, [13], which may render the classical notion of space-time obsolete in this case or replace it with another structure. An interesting framework in which the behavior of tensionless strings may be studied in detail is provided by the AdS/CFT correspondence when the gauge theory side becomes weakly coupled (see for instance, [12, 14, 15, 16, 17] and references therein). In any case, since very little is still known about the tensionless limit of string theory, and the symmetry breaking patterns of its structure, it is natural to expect that any further progress in this direction will be beneficial for the future development of the whole subject.

It is precisely this problem that we are going to address in the present work by considering the tensionless limit of some exactly solvable gauged WZW models at the quantum level. In these models, the tensionless limit arises group theoretically in the ultra-quantum region, which is well defined and tractable. The models makes good sense from the world-sheet point of view, although they invalidate all conventional effective field theory descriptions based on the α'-expansion. It will be shown that gravity decouples naturally from their spectrum in the form of a Liouville field with infinite background charge, but there is still a lot of structure that remains and can be treated in exact terms. Another advantage of these models is the non-trivial nature of their classical geometry, which receives substantial α' corrections within the usual perturbative expansion of the renormalization group equations, and they offer concrete examples for comparing the tensionless limits before or after quantization. However, we are still unable to provide a systematic reformulation of the complete theory in terms of $1/\alpha'$ expansion in target space, since this approach requires the introduction of new concepts and variables that are purely stringy in nature, without having an analogue in the language of conventional effective field theories. Finally, it should be emphasized that the tensionless WZW models do not necessarily arise as limiting cases within critical string theory, but this is not a drawback because the critical dimension is not a useful concept in the tensionless limit.

The supergravity description of string theory, which corresponds to the opposite (large tension) limit when $\alpha' \to 0$, provides a consistent truncation of string dynamics at low energy scales, where only the genuine massless modes participate, including the graviton and the dilaton, and the effective action consists of the usual Einstein terms, plus higher order curvature terms in the α'-expansion. The higher order terms are in principle calculable, but their determination is quite cumbersome unless new symmetry rules are invented using a more fundamental formulation of string dynamics that includes all α' corrections. This is precisely a place where the unbroken symmetry of tensionless strings, when appropriately described, may shed new light into the structure of all such higher order curvature terms. We note that the situation is reminiscent of the non-commutative structures that arise in the deformation approach to quantization: the non-commutative product admits a power series expansion in Planck's constant with higher derivative terms that obey consistency requirements, order by order, following from associativity. The computation of all deformation terms is made systematic once the notion of classical geometry is abandoned and one introduces operators acting on the Hilbert space, as the relevant concepts in the quantum theory, and declare that the non-commutative product of functions is isomorphic to the product of quantum operators. Furthermore, the use of operators provides the only way to treat systematically the ultra-quantum limit of non-commutative geometry, when Planck's constant tends to infinity, in which case there is no point to use power series expansions around the underlying classical concepts.

The tensionless limit we are considering in this paper is taken directly at the quantum level, which is most natural as large tension is related to large values of Planck's constant. Recall that α' appears as a loop counting parameter in the perturbative renormalization group analysis of the world-sheet sigma model, and the tensionless limit is ultra-quantum in nature with Planck mass equal to zero. Then, it is natural to expect that the reformulation of string theory in this case will lead to the introduction of new concepts that will also be valuable for finite values of α', in analogy with the operator approach to non-commutative geometry. It should also be added in this context that several problems of non-commutative field theories admit a simple formulation in the infinite non-commutativity limit, [18], whereas for finite values of the deformation parameter the treatment becomes more intricate. Thus, tensionless strings seem to offer the simplest framework in which new ideas can be brought to light. The WZW models provide a concrete framework in the attempt to link conformal field theories with non-commutative structures, following the general program outlined in reference [19], since the notion of classical geometry undergoes quantum deformations when the level of the underlying Kac-Moody algebras assume finite values, which are far away from the classical large k limit. Then, the tensionless limit that exists for non-compact models when the level k assumes its critical value, corresponds to infinitely large non-commutativity and the correspondence between strings and non-commutative structures becomes more pronounced.

The effective field theory description of tensionless strings seems to require the introduction of all massless states on equal footing, but such an enlarged action and its symmetries principles are not known for all string states at this moment. We have no idea about the target space framework that replaces Einstein gravity and its couplings to other

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fields when $\alpha' \to \infty$. However, we can develop an alternative route based on the world-sheet symmetries of the underlying two-dimensional quantum field theories that describe building blocks of string theory vacua, which also make good sense in the tensionless limit, as for any other value of the string tension. Typical examples are provided by gauged WZW models based on non-compact groups, such as the two-dimensional black-hole coset $\mathrm{SL}(2,\mathbb{R})_k/U(1)$ and higher dimensional generalizations thereof, which are well defined for all values of the central charge of the underlying Kac-Moody algebra that ranges from the dual Coxeter number to infinity.

The tensionless limit is reached when k approaches the dual Coxeter number, and it is well defined in the framework of two-dimensional quantum field theories. However, this limit is singular in the class of conformal field theories because the central charge of the Virasoro algebra becomes infinite and a rescaling of the Virasoro generators is required. In turn, this leads to a contraction of the conformal symmetry that amounts to decoupling gravity from the spectrum, but otherwise there is a large world-sheet symmetry that remains associated to higher spin currents. A primary aim of the present work is to expose the rich algebraic structures of these models, which arise as enhanced world-sheet symmetries when k assumes critical values. Put differently, the underlying Kac-Moody algebras have a large number of null states when their level becomes critical, and they are responsible for the exact treatment of WZW models in the tensionless limit. It should be noted, however, that this approach only applies to non-compact groups, since the compact models can never become critical by unitarity that restricts the allowed values of k.

Our work could be regarded as generalization of some original ideas introduced in reference [20], where non-compact WZW coset were put forward as models for tensionless strings. In this paper we carry out this program in great detail, using the target space and the world-sheet description of these models, and find that gravity decouples naturally from their spectrum. Using the two-dimensional black-hole coset $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$, as illustrative example, we find the exact metric becomes singular at $k=2$, which is consistent with the naive expectation that strings behave as tensionless objects in very strong gravitational fields. However, more careful analysis of the quantum tensionless limit reveals that gravity plays no role in this case, as it decouples in the form of a Liouville field with infinite background charge, and there is no remnant of space-time geometry. These models resemble "little string theories" which arise by taking the string coupling to zero in some configuration of Neveu-Schwarz five-branes and/or singularities, while keeping the string scale constant; for a review, see for instance [21], and references therein. Although different, they both define non-trivial theories which are decoupled from gravity.

The class of models we are considering here turn out to exhibit a rich symmetry structure in the tensionless limit, which is associated to higher spin fields and it is systematically described by a truncated version of the W_∞ algebra on the world-sheet of the resulting two-dimensional quantum theory. Quite remarkably, this algebra is linear and can be written in closed form using a bilinear realization of its generators in terms of a complex fermion. It should be noted, however, that it differs from the usual realizations of W_∞-type algebras, as there is no stress-energy tensor among its generators. It is conceivable that the tensionless limit of WZW models might also have a topological meaning that characterizes their behavior after the decoupling of gravity. Although this point of view will not be developed here, it may be closely related to the topological phase of tensionless strings in flat space that was advocated before, [13].

The remaining sections of the paper are organized as follows. In section 2, we briefly review the world-sheet and space-time description of gauged WZW models and discuss the exact form of the metric for all physical values of the level of the underlying Kac-Moody algebra. Special emphasis is placed on the (euclidean) two-dimensional black-hole coset $\mathrm{SL}(2,\mathbb{R})_k\,/\,\mathrm{U}(1)$, which is shown to exhibit tensionless limit for $k = 2$. In section 3, we discuss the spectrum of the $\mathrm{SL}(2,\mathbb{R})_k\,/\,\mathrm{U}(1)$ coset model, which consists of two parts coming from a Liouville field and a compactified boson, respectively, and describe the decoupling of the Liouville mode when $k = 2$. In section 4, we compare our results with the null gauging of WZW models, which correspond to an infinite boost in target space, and they describe the Liouville mode that decouples in the tensionless limit. In section 5, we employ the theory of non-compact parafermions to construct extended world-sheet symmetries of the coset $\mathrm{SL}(2,\mathbb{R})_k\,/\,\mathrm{U}(1)$, which include the Virasoro algebra for $2 < k < \infty$. The resulting algebraic structure is a non-linear deformation of W_∞, denoted by $\hat{W}_\infty(k)$, which linearizes in the large tension limit, $k \to \infty$. It is also shown that when k assumes its critical value, $k = 2$, $\hat{W}_\infty(k)$ also becomes linear and coincides with a truncated version of W_∞ generated by all integer higher spin fields with $s \geq 3$. In this case, the Virasoro algebra abelianizes by suitable rescaling of the generators, and it can be consistently factored out as it only depends on the Liouville mode that decouples in the tensionless limit. The precise identifications are made in section 6, where the $W_{1+\infty}$ algebra and its higher spin truncations are discussed in all generality; we also comment on their free field realizations as they arise from coset models. In section 7, we perform a formal BRST analysis of $\hat{W}_\infty(2)$ as (fundamental) world-sheet symmetry of tensionless gauged WZW models and show that the central charge of all higher spin generators should be fixed to a critical value, which is not seen by the contracted Virasoro symmetry. It turns out that this symmetry is not anomalous free for the black-hole coset at $k = 2$, but it needs two copies for consistent implementation. In section 8, we outline generalizations of the basic framework to higher dimensional coset models when the level of the underlying Kac-Moody algebras reach their critical values. Finally, in section 9, we present the conclusions and outline some directions for future work.

2. Gauged Wess-Zumino-Witten models

In this section we recall the classical description of gauged WZW models and summarize the group theoretical methods that allow to determine their exact geometry to all orders of $\alpha' \sim 1/k$. It is then possible to take the ultra-quantum limit by letting k assume its critical value, in order to establish the exact form of the very strong gravitational field which is responsible for the tensionless behavior of these models. The geometry becomes singular at $k = g^\vee$, as expected on general grounds, and conformal invariance can no longer be used within any perturbative renormalization group scheme to yield reliable space-time interpretation. The main focus is placed here on the euclidean black-

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hole coset $\mathrm{SL}(2,\mathbb{R})_k\,/\,\mathrm{U}(1)$, which provides the simplest gauged WZW model based on non-compact groups. Further examples with higher dimensional coset models will be included in section 8, where similar observations are also made.

2.1 Preliminaries

The WZW models, and their gauged versions, constitute exact conformal field theories, which are constructed group theoretically, and they can be used as building blocks for the description of string theory vacua. These models are based on the Kac-Moody symmetry of a group G that is generated by the singular part of the operator product expansion of the currents

$$J_A(z)J_B(w) = f_{AB}^{C}\frac{J_C(w)}{z-w} + \eta_{AB}\frac{k}{2(z-w)^2}\,, \tag{2.1}$$

where f_{AB}^C are the structure constants, η_{AB} is the standard metric, and k is the level of the current algebra (see, for instance, [22, 23, 24]). We will consider sigma models of the form G/H by gauging appropriately chosen subgroups of G with special emphasis on non-compact groups for which the level can vary continuously from the dual Coxeter number $g^\vee$ to infinity, i.e., $k \geq g^\vee$, in order to have unitarity. The dual Coxeter number is given by the value of the quadratic Casimir operator of G in the adjoint representation, so that

$$g^\vee \eta^{AB} = f^{ACD}f^{B}{}_{CD}\,. \tag{2.2}$$

The non-compact cosets are most appropriate for constructing exact tensionless models by taking the limit $k \to g^\vee$, whereas their classical geometry corresponds to the limit $k \to \infty$. Also, in this context, one may extrapolate continuously between the two limits, since the gauged WZW models are well defined two-dimensional quantum field theories for all such values of k. Compact groups do not allow for this possibility because their level is quantized and it can never become critical, i.e., equal to the dual Coxeter number, while maintaining unitarity; this can be also formally seen by changing the sign of k in order to pass to the compact group, in which case the corresponding level is positive and it can never become equal to $-g^\vee$.

The conformal symmetry of WZW models G_k is realized by the Sugawara construction of their stress-energy tensor,

$$T(z) = \frac{1}{k-g^\vee}\eta^{AB}J_AJ_B(z)\,, \tag{2.3}$$

where normal ordering of the Kac-Moody currents is implicitly assumed. Then, the operator product expansion

$$T(z)T(w) = \frac{\partial T(w)}{z-w} + 2\frac{T(w)}{(z-w)^2} + \frac{c}{2(z-w)^4} \tag{2.4}$$

generates the Virasoro algebra with central charge

$$c_G = \frac{(\dim G)k}{k-g^\vee}\,. \tag{2.5}$$

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The stress-energy tensor of the gauged WZW models G/H is simply provided by the formula, [22, 23],

$$T_{G/H}(z) = T_G(z) - T_H(z) \tag{2.6}$$

and the corresponding central charge equals to the difference of the two individual terms,

$$c_{G/H} = c_G - c_H = \frac{(\dim G)k}{k - g^\vee} - \frac{(\dim H)k}{k - h^\vee}\,, \tag{2.7}$$

where $g^\vee$ and $h^\vee$ are the dual Coxeter numbers of G and H, respectively. Thus, we observe that $c_{G/H} \to \dim(G/H)$ when $k \to \infty$, whereas $c_{G/H} \to \infty$ when $k \to g^\vee$. This is precisely the value of interest in the tensionless limit, since $k - g^\vee \sim 1/\alpha'$. Better understanding of this relation will be achieved later using the effective action of the coset conformal field theories.

The fact that the central charge of the Virasoro algebra becomes infinite implies that the conformal field theory of the coset model is singular at $k = g^\vee$. Appropriate rescaling of the Virasoro generators is then required in order to make the coefficient of the central terms finite, in which case the conformal algebra contracts to an abelian structure.[1] This issue will be discussed later in great detail. We will find that the tensionless limit makes perfect sense as two-dimensional quantum field theory, although it is singular as conformal field theory.

2.2 Classical considerations

The starting point is the ordinary WZW model for a Lie group G, which is defined by the action, [25],

$$S_{\rm WZW} = \frac{k}{4\pi}\int_\Sigma d^2z\, {\rm Tr}\left(\partial g^{-1}\bar\partial g\right) + \frac{ik}{24\pi}\int_B d^3x\, {\rm Tr}\left(g^{-1}dg\right)^3\,. \tag{2.8}$$

Here, $(z, \bar z)$ are complex coordinates on the two-dimensional world-sheet Σ, whereas the second term is topological and it is defined on a three-dimensional manifold B whose boundary is Σ; for all practical purposes Σ is taken to be a sphere and B is a three-dimensional ball. This action has global $G \times G$ symmetry corresponding to $g \to agb^{-1}$ with both $a, b \in G$. One may also consider a variant of the WZW models by gauging a subgroup of the global symmetry group, but this is not always possible unless the subgroup obeys a certain anomaly cancellation condition. In the following we consider the gauging of anomaly free subgroups for the case of non-compact simple Lie groups G.

The gauging of the WZW model with respect to a subgroup $H \subset G$ is implemented by introducing gauge fields A and $\bar A$ with values in the Lie algebra of H, and the action is taken to be [25, 26, 27]

$$S(g; A, \bar A) = S_{\rm WZW} - \frac{k}{2\pi}\int d^2z\, {\rm Tr}\left(A\bar\partial g g^{-1} - \bar A g^{-1}\partial g - Ag\bar A g^{-1} + A\bar A\right). \tag{2.9}$$

[1]This contraction is similar to the familiar Inonü-Wigner contraction of Lie algebras; for example, the SU(2) algebra contracts to the Heisenberg-Weyl algebra in the infinite spin limit, since a sphere with infinite radius looks like a two-dimensional plane.

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$S(g; A, \bar{A})$ is invariant under the local gauge transformations

$$A \to h^{-1}(\partial + A)h\,, \qquad \bar{A} \to h^{-1}(\bar{\partial} + \bar{A})h\,, \tag{2.10}$$

with g also transforming in a vector-like way, as

$$g \to hgh^{-1}\,; \qquad h \in H\,. \tag{2.11}$$

The classical equations of motion that follow by variation with respect to all fields are

$$\delta A: \;\; \bar{D}gg^{-1}\,|_H = 0\,, \tag{2.12}$$

$$\delta \bar{A}: \;\; g^{-1}Dg\,|_H = 0\,, \tag{2.13}$$

$$\delta g: \;\; \bar{D}\left(g^{-1}Dg\right) + F_{z\bar{z}} = 0\,, \tag{2.14}$$

where $F_{z\bar{z}} = \partial\bar{A} - \bar{\partial}A + [A, \bar{A}]$ is the field strength and D, $\bar{D}$ are the corresponding covariant derivatives. Then, imposing the condition (2.13) on equation (2.14), one arrives at the zero curvature condition $F_{z\bar{z}} = 0$, and

$$\bar{D}\left(g^{-1}Dg\right)|_{G/H} = 0\,. \tag{2.15}$$

We may use appropriate parametrization of the group elements $g(z, \bar{z})$ to fix the gauge freedom (2.11) and integrate over the gauge fields A, $\bar{A}$ in order to obtain the effective action of the gauged WZW model for the coset G/H. This procedure can be equivalently implemented at the classical level by first solving for the gauge fields, which act as Lagrange multipliers, and then substitute the resulting expressions in terms of $g(z, \bar{z})$ back into the action $S(g; A, \bar{A})$, [26, 27, 28]. In particular, choosing a unitary gauge in the fundamental representation of G, we may first fix $\dim H$ variables among the total number of $\dim G$ parameters of the group elements g, and denote by X^μ the remaining $\dim(G/H)$ target space variables. Then, the gauge fields can be eliminated from the action $S(g; A, \bar{A})$, using their equations of motion

$$\begin{aligned} A^a &= +i\left(C^T - I\right)^{-1}_{ab} L^b_\mu \partial X^\mu\,, \\ \bar{A}^a &= -i\left(C - I\right)^{-1}_{ab} R^b_\mu \bar{\partial} X^\mu\,. \end{aligned} \tag{2.16}$$

The indices of the Lie algebra G split as $A = (a, \alpha)$ with $a \in H$ and $\alpha \in G/H$ and denote the generators of H by T^a. Also, we adopt the following short-hand notations

$$L^a_\mu = -i\,\mathrm{Tr}\left(T^a g^{-1}\partial_\mu g\right)\,, \qquad R^a_\mu = -i\,\mathrm{Tr}\left(T^a \partial_\mu g g^{-1}\right)\,, \qquad C^{ab} = \mathrm{Tr}\left(T^a g T^b g^{-1}\right)\,. \tag{2.17}$$

Finally, the sigma model action of the gauged WZW model is written in terms of these variables as follows,

$$S = S_{\mathrm{WZW}}(g) - \frac{k}{2\pi}\int d^2z R^a_\mu \left(C^T - I\right)^{-1}_{ab} L^b_\nu \partial X^\mu \bar{\partial} X^\nu\,. \tag{2.18}$$

In any case, the target space metric depends only on the gauge invariant parameters that are left to parametrize g after exploiting the gauge freedom (2.11); in general, there is also an anti-symmetric tensor field that originates from the topological term of the action $S_{\rm WZW}(g)$. The only extra ingredient that requires special attention in the quantum theory is the introduction of a target space dilaton field due to finite corrections coming from the integration over the gauge fields. The dilaton should be added in the effective action, in the usual way, in order to maintain conformal invariance of the model to lowest order in $\alpha' \sim 1/k$, as $k \to \infty$. Higher order corrections modify the form of the background fields and they also provide the exact relation between the level k and the tension parameter of these models.

The two commuting copies of the Kac-Moody algebra that correspond to the chiral sectors of the WZW model G_k have remnants in the gauged WZW coset, and they are associated to the (so called) parafermion currents, [29, 28, 30]. They are classically defined by first parametrizing the gauge fields as pure gauge, i.e., $A = -\partial h h^{-1}$ and $\bar{A} = -\bar{\partial}\bar{h}\bar{h}^{-1}$ in terms of appropriately chosen group elements h, $\bar{h} \in H$. Then, introducing the gauge invariant element $f = h^{-1} g h$ and using the zero curvature condition $F_{z\bar{z}} = 0$, the classical equations of motion (2.15) are written as chiral conservation laws, $\bar{\partial}\Psi = 0$, where the field

$$\Psi(z) = \frac{ik}{\pi} f^{-1}\partial f(z) \tag{2.19}$$

defines the classical parafermion current with values in the coset space G/H. Likewise, the anti-holomorphic parafermion current $\bar{\Psi}$ is defined using the group elements $\bar{h}$.

It is important to realize in this context that the parafermion currents are non-local fields, since they have Wilson lines attached to them which arise by solving h and $\bar{h}$ in terms of A and $\bar{A}$, respectively, using path-ordered exponentials. Their classical Poisson algebra is computed using the coset valued matrix elements Ψ_α, and it substitutes for the current symmetry algebra of the G_k WZW models. In the quantum theory, this algebra corresponds to the singular part of the operator product expansion of the parafermion currents, [29], although non-singular terms are also important in order to define W-algebra generators of the extended conformal symmetries of these cosets. Later, we will pursue this method directly in the quantum theory for arbitrary values of the level k, and examine the algebraic structures that result on the world-sheet when k comes close to its critical value, $g^\vee$. In this framework, we will be able to determine the exact properties of the two-dimensional quantum field theories that correspond to the tensionless limit of the non-compact WZW models.

We also note for completeness that apart from vector gauging, it is also possible to perform axial gauging when H is an anomaly free *abelian* subgroup. In this case, the starting point is provided by the action

$$S'(g; A, \bar{A}) = S_{\rm WZW} - \frac{k}{2\pi}\int d^2z \, {\rm Tr}\left(A\bar{\partial}g g^{-1} + \bar{A}g^{-1}\partial g + A g \bar{A} g^{-1} + A\bar{A}\right), \tag{2.20}$$

which is invariant under the axial-like local gauge transformations

$$g \to hgh\,; \qquad h \in H\,, \tag{2.21}$$

whereas A and $\bar{A}$ transform as in the vector gauging. The rest proceeds as before, but the computation yields different background fields in target space which is T-dual to the geometry of vector gauging, [31].

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The simplest example is provided by the choice $G = \mathrm{SL}(2,\mathbb{R})$ with H being an abelian subgroup, [32]. One possibility corresponds to the non-compact abelian subgroup generated by the third Pauli matrix σ_3 in the fundamental representation of $\mathrm{SL}(2,\mathbb{R})$. Then, the coset model is $\mathrm{SO}(2,1)_k/\,\mathrm{SO}(1,1) \simeq \mathrm{SL}(2,\mathbb{R})_k/R$ with lorentzian signature and it describes the classical geometry of a two-dimensional black-hole in the large k limit. The other possibility corresponds to gauging the compact abelian subgroup generated by $i\sigma_2$ in terms of the second Pauli matrix. It leads to the coset model $\mathrm{SO}(2,1)_k/\,\mathrm{SO}(2) \simeq \mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$ that describes the geometry of a euclidean black-hole. In both cases the gauging can be implemented either vectorially or axially, since H is abelian, and the resulting backgrounds are related to each other by T-duality. Also, the lorentzian and euclidean black-holes are naturally related to each other by analytic continuation of their target space coordinates.

The derivation of the explicit expressions is quite standard and we refer the reader to the original works for further details, [32, 33, 34]. Besides, in the next subsection, the exact metric and dilaton fields are presented systematically to all orders in $\alpha' \sim 1/k$, and the classical geometry of the cosets follows when $k \to \infty$. Higher dimensional cosets will be discussed briefly in section 8, but their algebraic and geometric structures become quickly rather involved, and hence difficult to treat in all generality. In any case, the two-dimensional black-hole coset provides a good laboratory for understanding the behavior of gauged WZW models at critical level. Most of our analysis will be subsequently confined to applications to the euclidean black-hole coset.

2.3 Exact metric and the tensionless limit

Next, we consider the quantum modifications to the classical geometry, which are induced by adding $\alpha' \sim 1/k$ corrections to the target space fields using the perturbative beta function equations. Fortunately, the quantum corrections can be explicitly worked out in all WZW models by appealing to different (but equivalent) methods for the construction of the exact metric and other background fields to all orders in α'. Then, one may formally take the tensionless limit of the exact formulae in order to get a feeling of the resulting geometry in the ultra-quantum regime. This method also provides the exact relation between k and α' beyond the leading order approximation. In all cases the geometry becomes singular in the limit $\alpha' \to \infty$, which agrees with the naive expectation that strings behave as tensionless objects in strong gravitational fields near the singularities, [7]. However, as we will see later, a careful analysis of the quantum theory shows that gravity decouples in the tensionless limit of WZW models and there is no remnant of the target space geometry: the theory is non-trivial but non-geometric.

The simplest way to derive the exact form of the metric is provided by the hamiltonian approach, which asserts that the Laplace operator in target space is given by $L_0 + \bar{L}_0$ in terms of the left and right-moving Virasoro zero modes. In particular, using the effective action for the tachyon field T, it follows that

$$L_0 T = \left(\frac{\Delta_G^L}{k - g^\vee} - \frac{\Delta_H^L}{k - h^\vee} \right) T \,, \tag{2.22}$$

where Δ_G^L and Δ_H^L are the quadratic Casimir operators of the groups G and H, respectively. This equation follows from the Sugawara construction of the coset model G/H, and there is also a similar action for the operator $\bar{L}_0$ in terms of the operators Δ_G^R and Δ_H^R. It can be shown in all generality that $\Delta_G^L = \Delta_G^R$, whereas $(\Delta_H^L - \Delta_H^R)T = 0$ is only valid on-shell when acting on the tachyon field. This is also consistent with the gauge invariance condition $(J_H^L + J_H^R)T = 0$ which is imposed on the tachyon field by the vector gauging of the models; for the axial gauging this condition is replaced by $(J_H^L - J_H^R)T = 0$. Then, the target space metric and dilaton fields are chosen so that the exact hamiltonian acts in the following way,

$$(L_0 + \bar{L}_0)T = -\frac{e^{-\Phi}}{\sqrt{G}}\partial_i\left(e^{\Phi}\sqrt{G}G^{ij}\partial_j T\right). \tag{2.23}$$

An invariant expression is $\sqrt{G}\exp\Phi$, which is independent of k.

This identification was first suggested in reference [35], and it was subsequently applied by a number of authors to a variety of conformal field theory models, [36, 37, 38]. The resulting expressions for the background fields have been tested extensively by comparison to the perturbative expansion of the beta functions equations to higher orders in α'. There are also independent derivations based on the effective action, which can be made systematic by the exact solvability of these models; for an excellent exposition of all different approaches see, for instance, [38]. The quantum analysis also leads to the identification, in appropriate units[2]

$$\alpha' = \frac{1}{k - g^{\vee}}, \tag{2.24}$$

which follows by replacing k with $k - g^{\vee}$ in all cases.

The tensionless limit is reached when $k = g^{\vee}$, according to (2.24), but conformal invariance can no longer be used to yield reliable space-time interpretation of these models. This method suggests the way to take the tensionless limit in the framework of the effective field theory, but the target space geometry becomes highly singular. The appearance of singularities implies that the sigma model description breaks down in this case as all other massless states should be included on equal footing, if possible. Consequently, the tensionless limit of the theory cannot be addressed systematically in the present framework, which is inadequate as it stands. It is only considered here to establish the form of the singularities, which break the validity of the sigma model approach to string theory, and compare with other methods.

Next, we focus on the exact form of the metric and dilaton fields for the simplest euclidean black-hole coset and postpone generalizations to higher rank spaces until section 8. Since $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ can be constructed by gauging the $\mathrm{U}(1)$ subgroup in two different ways, the above hamiltonian procedure should be applied separately to the geometry of the momentum and winding modes. In either case, the same qualitative picture results when $k = 2$, namely that the geometry looks like an infinitely curved hyperboloid written

[2] The coordinates X^{μ} of non-linear sigma models are dimensionful but they can be rescaled by the characteristic radius of the manifold R to dimensionless fields. The WZW action is written in terms of the group elements g so that the level $k = R^2/\alpha'$ is the dimensionless analogue of the string tension. Classically, the tensionless limit arises when $k \to 0$, but quantum mechanically one has to include the shift by $g^{\vee}$.

in different coordinate patches that depend on the gauging. The intermediate results are quite standard by now, but they are summarized below following [36], in order to examine the special limit $k = 2$, which is of interest here. Also, they will be used to discuss the spectrum in section 3 that relies on the same hamiltonian method.

Using the standard parametrization of the $\mathrm{SL}(2, \mathbb{R})$ group elements

$$g = \exp\left(\frac{i}{2}\theta_{\mathrm{L}}\sigma_2\right) \exp\left(\frac{1}{2}r\sigma_1\right) \exp\left(\frac{i}{2}\theta_{\mathrm{R}}\sigma_2\right) \tag{2.25}$$

in terms of Pauli matrices, we may work out the laplacian of the exact sigma model. First, consider the operator $L_0 = L_0^{\mathrm{SL}(2,\mathbb{R})} - L_0^{\mathrm{U}(1)}$, where the individual terms are given by the Sugawara construction in terms of the Fourier modes,

$$\begin{aligned} L_0^{\mathrm{SL}(2,\mathbb{R})} &= -\frac{1}{k-2}\left(\mathcal{C}_2 + \sum_{n=1}^{\infty}\left(J_{-n}^{+}J_n^{-} + J_{-n}^{-}J_n^{+} + 2J_{-n}^{3}J_n^{3}\right)\right), \\ L_0^{\mathrm{U}(1)} &= -\frac{1}{k}\left(\left(J_0^3\right)^2 + 2\sum_{n=1}^{\infty} J_{-n}^{3}J_n^{3}\right). \end{aligned} \tag{2.26}$$

Here, $\mathcal{C}_2$ denotes the Casimir operator given by the quadratic expression of zero modes,

$$\mathcal{C}_2 = \frac{1}{2}\left(J_0^{+}J_0^{-} + J_0^{-}J_0^{+}\right) + \left(J_0^3\right)^2 \tag{2.27}$$

There are also similar expressions for the operators $\bar{L}_0^{\mathrm{SL}(2,\mathbb{R})}$ and $\bar{L}_0^{\mathrm{U}(1)}$ in terms of the corresponding anti-holomorphic currents.

Note that only the zero modes $J_0^{\pm}$ and J_0^3 contribute to the action of the operators on the tachyon field, since the action of the positive modes on highest weight states give zero. Therefore, it is sufficient to use the differential form of the global $\mathrm{SL}(2, \mathbb{R})$ generators in order to represent the relevant part of the operator L_0. Since

$$J_0^{\pm} = e^{\mp i\theta_{\mathrm{L}}}\left(\frac{\partial}{\partial r} \pm \frac{i}{\sinh r}\left(\frac{\partial}{\partial\theta_{\mathrm{R}}} - \cosh r\frac{\partial}{\partial\theta_{\mathrm{L}}}\right)\right), \qquad J_0^3 = i\frac{\partial}{\partial\theta_{\mathrm{L}}}, \tag{2.28}$$

the action of L_0 and $\bar{L}_0$ on the tachyon field T is given by the differential operators

$$L_0 = -\frac{\Delta_0}{k-2} - \frac{1}{k}\frac{\partial^2}{\partial\theta_{\mathrm{L}}^2}, \qquad \bar{L}_0 = -\frac{\Delta_0}{k-2} - \frac{1}{k}\frac{\partial^2}{\partial\theta_{\mathrm{R}}^2}, \tag{2.29}$$

respectively, where Δ_0 is

$$\Delta_0 = \frac{\partial^2}{\partial r^2} + \coth r\frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\left(\frac{\partial^2}{\partial\theta_{\mathrm{L}}^2} + \frac{\partial^2}{\partial\theta_{\mathrm{R}}^2} - 2\cosh r\frac{\partial^2}{\partial\theta_{\mathrm{L}}\partial\theta_{\mathrm{R}}}\right). \tag{2.30}$$

Clearly, Δ_0 is invariant under the interchange $L \leftrightarrow R$, as expected on general grounds. These expressions are also particularly useful for computing the spectrum of conformal dimensions in the black-hole coset. Furthermore, the condition $(L_0 - \bar{L}_0)T = 0$ implies the separation of variables $T = T(r, \theta) + \tilde{T}(r, \tilde{\theta})$ where

$$\theta = \frac{1}{2}(\theta_{\mathrm{L}} - \theta_{\mathrm{R}}), \qquad \tilde{\theta} = \frac{1}{2}(\theta_{\mathrm{L}} + \theta_{\mathrm{R}}), \tag{2.31}$$

and it leads to different effective descriptions of the geometry of momentum and winding modes.

Using the variable θ, which arises in the study of the momentum modes, the action of the laplacian on $T(r, \theta)$ is given by the differential operator

$$L_0 = -\frac{1}{k-2}\left(\frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \left(\coth^2 \frac{r}{2} - \frac{2}{k}\right)\frac{\partial^2}{\partial \theta^2}\right) . \tag{2.32}$$

The exact expressions for the effective metric and dilaton fields follow by comparison with the Laplace operator (2.23), and they assume the special form

$$ds^2 = \frac{1}{2}(k-2)\left(dr^2 + \beta^2(r) d\theta^2\right) , \qquad \Phi = \log\left(\frac{\sinh r}{\beta(r)}\right) \tag{2.33}$$

for all $k \geq 2$, where $\beta(r)$ is given by the function

$$\frac{4}{\beta^2(r)} = \coth^2 \frac{r}{2} - \frac{2}{k} . \tag{2.34}$$

The results simplify considerably when k approaches 2 or infinity, leading to the following geometric structures: First, as $k \to \infty$, the geometry in the gravitational regime looks like

$$ds^2 \simeq \frac{1}{2} k \left(dr^2 + 4\tanh^2 \frac{r}{2} d\theta^2\right) , \qquad \Phi \simeq 2\log\left(\cosh \frac{r}{2}\right) , \tag{2.35}$$

and describes the familiar semi-infinite cigar when the θ coordinate is chosen to be periodic modulo 2π. As such, it satisfies the conditions for conformal invariance to lowest order in α'; they read as $R_{\mu\nu} = \nabla_\mu \nabla_\nu \Phi$ fixing the normalization of the dilaton field. Second, continuing the validity of the exact solution close to the critical level of the underlying $\mathrm{SL}(2, \mathbb{R})_k$ algebra, we find

$$ds^2 \simeq \frac{1}{2}(k-2)\left(dr^2 + 4\sinh^2 \frac{r}{2} d\theta^2\right) , \qquad \Phi \simeq \log\left(\cosh \frac{r}{2}\right) , \tag{2.36}$$

which describes the geometry of an infinitely curved hyperboloid in appropriate coordinates. In this case, there is also a dilaton field that accompanies the geometry, which is everywhere regular in space and the string coupling $\exp(-\Phi)$ never becomes infinite. It provides a "target space" realization of the tensionless $\mathrm{SL}(2, \mathbb{R})_2 / \mathrm{U}(1)$ coset model and establishes the singular nature of its effective geometry in the ultra-quantum regime. However, conformal invariance can not be reliably used in this case because the beta function equations are only valid perturbatively in $\alpha' \simeq 1/k$ when k is large, and the result should be interpreted with care.

Likewise, the geometry of the winding modes follows by duality transformation of the $\mathrm{SL}(2, \mathbb{R})$ currents, $J^a \to J^a$ and $\bar{J}^a \to -\bar{J}^a$, which relate the axial with the vector gauging. The duality interchanges the angular variables $\theta \leftrightarrow \tilde{\theta}$ and the laplacian on $\tilde{T}(r, \tilde{\theta})$ is represented by the differential operator

$$L_0 = -\frac{1}{k-2}\left(\frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \left(\tanh^2 \frac{r}{2} - \frac{2}{k}\right)\frac{\partial^2}{\partial \tilde{\theta}^2}\right) . \tag{2.37}$$

The relevant expressions for the effective metric and dilaton fields are also put in the form (2.33), using another function

$$\frac{4}{\beta^2(r)} = \tanh^2 \frac{r}{2} - \frac{2}{k} . \tag{2.38}$$

The two cases are formally related to each other by the simple transformation rule $r \to r + i\pi/2$ and $\theta \to \tilde{\theta}$. However, the metric and dilaton fields are now singular at $r = r_0 = 2\,\mathrm{arctanh}\,\sqrt{2/k}$. This critical value defines the range of validity of the metric, since the signature changes beyond it: for $r > r_0$ the signature is $++$, whereas for $r < r_0$ the signature changes to $+-$.

As before, there are two distinct limits corresponding to values of k close to 2 or infinity. In the gravitational regime, $k \to \infty$, the background fields are

$$ds^2 \simeq \frac{1}{2}k\left(dr^2 + 4\coth^2\frac{r}{2}d\tilde{\theta}^2\right), \qquad \Phi \simeq 2\log\left(\sinh\frac{r}{2}\right), \tag{2.39}$$

satisfying the conditions for conformal invariance to lowest order in α'. This geometry is a trumpet with curvature singularity at the origin of the coordinate system, and it is dual to the semi-infinite cigar. On the other hand, when $k \to 2$, it follows that

$$ds^2 \simeq \frac{1}{2}(k-2)\left(dr^2 - 4\cosh^2\frac{r}{2}d\tilde{\theta}^2\right), \qquad \Phi \simeq \log\left(\sinh\frac{r}{2}\right), \tag{2.40}$$

which provides the "target space" description of the tensionless $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ model in terms of an infinitely curved AdS_2 space with signature $+-$. Note that the change of signature occurs because the critical value r_0 is pushed to infinity when $k \to 2$. Thus, either we are prepared to accept a change of signature in space, in which case the string coupling $\exp(-\Phi)$ becomes infinite at the origin, $r = 0$, but it is regular everywhere else, or else the available space covered by the coordinate system is completely eaten up to maintain euclidean signature. In either case, the singular character of the exact metric indicates, as before, that the effective theory breaks down in the tensionless limit and conformal invariance is not particularly useful for exploring the geometry of target space beyond perturbation theory.

The lorentzian version of the coset arises by analytic continuation of the angular variable, so that r remains a spatial coordinate, and it corresponds to the gauged WZW model $\mathrm{SL}(2,\mathbb{R})_k/R$. Then, the tensionless limit of the two-dimensional geometry is AdS_2 with zero radius, using the axial gauging. On the other hand, vector gauging leads to a confusion of signature, as before, and it gives rise to an infinitely curved hyperboloid with euclidean signature or else all space is eaten up by the quantum corrections. This coincidence offers a concrete framework for exploring the relation between the quantum tensionless limit of the black-hole coset at $k = 2$ and the quantization of classical tensionless strings on AdS_2, following the light-cone methods of references [16, 17] (but see also [6]). It should be further mentioned that the higher dimensional WZW models $\mathrm{SO}(d-1,2)/\mathrm{SO}(d-1,1)$ with lorentzian signature do not represent AdS_d space with zero radius when their effective action is taken to critical level. As we will see later in section 8, they rather describe non-symmetric deformations of AdS_d for $d > 2$, in the presence of non-trivial anti-symmetric tensor field, but this may not be so important in the zero radius limit. These models also exhibit a confusion of signature or else there is a truncation of the available space, as for the vector gauged $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ coset. In any case, the theory of non-compact cosets provides exact models for tensionless strings, which could be used further to understand the subtle issues of the AdS/CFT correspondence in the zero radius limit.

Summarizing the results of this exposition, we conclude that the metric sector of gauged WZW models becomes highly singular at critical k and string propagation behaves as tensionless theory. Since the effective field theory description breaks down, other methods should be employed for the exact quantum mechanical treatment of WZW models in this case. These will be provided later using world-sheet techniques which make perfect sense in the tensionless limit. In order to motivate some of the constructions, we will first discuss the behavior of the spectrum close to critical level and observe a decoupling of gravity from the remaining fields of the quantum theory.

3. Spectrum of the $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ coset

We briefly review the construction of the full spectrum of conformal dimensions for the $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ model, following [36, 39]. The results are analogous to the spectrum of the compact coset $\mathrm{SU}(2)_N/\mathrm{U}(1)$, but with some additional elements for the non-compact group. Then, we discuss the rescaling which is necessary to make sense of the special limit $k = 2$, and observe that gravity decouples from the spectrum in the form of a Liouville field with infinite background charge. The same picture will arise later using world-sheet methods, as the Virasoro algebra decouples from all remaining symmetries of the coset model.

3.1 The spectrum for $k > 2$

The spectrum of primary fields of the $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ model can be determined from the states of the $\mathrm{SL}(2,\mathbb{R})_k$ conformal field theory by imposing restrictions on the left and right moving J^3-oscillators,

$$J_n^3|\mathrm{state}\rangle = 0 = \bar{J}_n^3|\mathrm{state}\rangle \qquad \text{for } n > 0\,, \tag{3.1}$$

which are supplemented by the following conditions on the zero modes,

$$J_0^3 - \bar{J}_0^3 = m\,, \qquad J_0^3 + \bar{J}_0^3 = nk \tag{3.2}$$

for all integers m, n. Using the parametrization (2.25) of the $\mathrm{SL}(2,\mathbb{R})$ group elements, the condition (3.2) translates into the restriction

$$\omega_{\mathrm{L}} = \frac{1}{2}(m + nk)\,, \qquad \omega_{\mathrm{R}} = -\frac{1}{2}(m - nk) \tag{3.3}$$

for the lattice of the corresponding $\mathrm{U}(1)$ quantum numbers. Then, the full spectrum of conformal dimensions of the coset model can be obtained by diagonalizing the operators $L_0 = L_0^{\mathrm{SL}(2,\mathbb{R})} - L_0^{\mathrm{U}(1)}$ and $\bar{L}_0 = \bar{L}_0^{\mathrm{SL}(2,\mathbb{R})} - \bar{L}_0^{\mathrm{U}(1)}$.

The action of the $\mathrm{SL}(2,\mathbb{R})_k$ currents is computed by taking into account only the contribution of the zero modes $J_0^{\pm}$ and J_0^3, whereas the action of positive modes on highest weight states gives zero. Thus, it is sufficient to use the global $\mathrm{SL}(2,\mathbb{R})$ generators in order to represent the relevant part of the operators L_0 and $\bar{L}_0$, as in equation (2.29) in terms of the Casimir $\mathcal{C}_2$. Using the defining relations of highest weight states for both left and right Virasoro movers,

$$L_0|l,\omega_{\rm L}\rangle = h_l^{\omega_{\rm L}}|l,\omega_{\rm L}\rangle\,, \qquad \bar{L}_0|l,\omega_{\rm R}\rangle = \bar{h}_l^{\omega_{\rm R}}|l,\omega_{\rm R}\rangle\,, \tag{3.4}$$

the corresponding conformal weights are

$$\begin{aligned} h_l^{\omega_{\rm L}} &= -\frac{l(l+1)}{k-2} + \frac{1}{k}\omega_{\rm L}^2 = -\frac{l(l+1)}{k-2} + \frac{(m+nk)^2}{4k}\,, \\ \bar{h}_l^{\omega_{\rm R}} &= -\frac{l(l+1)}{k-2} + \frac{1}{k}\omega_{\rm R}^2 = -\frac{l(l+1)}{k-2} + \frac{(m-nk)^2}{4k}\,. \end{aligned} \tag{3.5}$$

Here, l is the $\mathrm{SL}(2,\mathbb{R})$ isospin which is determined by the quadratic Casimir operator (2.27) with eigenvalues $c_2 = l(l+1)$.

The allowed values of l follow from the classification of the unitary irreducible representations of the Lie algebra $\mathrm{SL}(2,\mathbb{R})$, supplemented by some additional restrictions that depend on the central charge k of the Kac-Moody algebra. Recall at this point that the representations of the global $\mathrm{SL}(2,\mathbb{R})$ algebra fall into three general series according to the allowed values of the isospin l and the "magnetic" quantum number ω that label the eigenvalues of $\mathcal{C}_2$ and the $\mathrm{U}(1)$ generators, respectively:

(a) *Principal continuous series*, which have $l = is - 1/2$ with real values of s and ω.

(b) *Complementary continuous series*, which have real $l \in [-1, -1/2]$ and ω is also real.

(c) *Principal discrete series*, which have real $l < -1/2$ and $|\omega| + l$ is non-negative integer.

The discrete series come in two different types, either highest or lowest weight, and there is also the trivial representation with $l = 0$ and $\omega = 0$ that should be added for completeness. Note that the quadratic Casimir is always real with $-l(l+1)$ being positive for the continuous series representations.

The only relevant representations turn out to be the (a) or (c) series leading to normalizable vertex operators. For the discrete series, both $|\omega_{\rm L,R}| + l$ should be non-negative integers, which in turn put restrictions on the allowed range of l depending on the level k. Taking also into account recent analysis based on spectral flows, [39, 40], one finds that the allowed range of values for the discrete series is

$$-\frac{1}{2}(k-1) < l < -\frac{1}{2} \tag{3.6}$$

in order to have unitarity. There is no such restriction on the principal continuous series. Thus, we obtain a complete description of the spectrum (3.5) in either case. It is also important to stress at this point that there are no discrete representations appearing at $k = 2$, using the unitarity bound (3.6).

As $k \to 2$, only the principal continuous series representations become relevant with

$$h_l^{\omega_{\rm L}} = \frac{s^2 + 1/4}{k-2} + \frac{(m+nk)^2}{4k}\,, \qquad \bar{h}_l^{\omega_{\rm R}} = \frac{s^2 + 1/4}{k-2} + \frac{(m-nk)^2}{4k}\,, \tag{3.7}$$

but the first term blows up in the limit. Its contribution will be attributed to a Liouville field that arises in the free field representation of the coset model and produces the same spectrum as above. Also, in section 4, the same Liouville theory will control the effective description of the $\mathrm{SL}(2,\mathbb{R})_k / \mathrm{U}(1)$ coset model, which is obtained by introducing a boost with very high Lorentz factor in the Lie algebra. This Liouville theory describes the radial coordinate of the coset model and has to decouple at critical level; otherwise, the dimensions (3.7) will contain an infinite part, which is natural to expect at very high energy scales but not in tensionless models. Put differently, unless the gravitational effects of Liouville theory can be consistently removed, the high energy limit cannot be considered as being tensionless. This crucial point will be clarified further in the following for the gauged and the ungauged WZW models. It is also interesting to recall that the spectral flow of the continuous representations correspond to long string states in the $\mathrm{SL}(2,\mathbb{R})_k$ WZW theory, [39, 41]. On the other hand, the discrete representations and their spectral flow correspond to short strings, but they do not arise at $k = 2$. Since the radial coordinate of long strings is effectively described by a Liouville theory with background charge $Q \sim 1/\sqrt{k-2}$, as follows from the exact analysis of the problem in the context of the $D1/D5$ brane system, [41], one is lead to suspect that their radial dynamics decouples in the tensionless limit.

3.2 The decoupling of Liouville field at $k = 2$

Next, we examine the reductions that arise at critical level of the $\mathrm{SL}(2,\mathbb{R})_k$ current algebra. The conformal dimensions (3.7) consist of two terms that behave differently when $k \to 2$; as they stand, the first blows up like $1/(k-2)$ and the second remains finite. Likewise, the central charge of the Virasoro algebra blows up like $1/(k-2)$. Thus, a rescaling is required in order to make sense of these infinities in a systematic way.

It is instructive for this purpose to describe the spectrum of conformal dimensions (3.5) for arbitrary level k using another conformal field theory

$$S = \frac{1}{2\pi} \int d^2 z \left(\partial\phi_1 \bar{\partial}\phi_1 + \partial\phi_2 \bar{\partial}\phi_2 + \frac{Q}{4}\sqrt{g} R \phi_1 \right) \tag{3.8}$$

that contains two free scalar fields, one of them with background charge Q and the other compactified on a circle with radius R. The field ϕ_2 has stress-energy tensor

$$T_2(z) = -\frac{1}{2}(\partial\phi_2)^2(z) \tag{3.9}$$

with Virasoro central charge $c = 1$ and the full mass spectrum consists of states with dimensions

$$M^2 = \frac{1}{2}\left(\frac{m}{2R} + nR\right)^2 , \tag{3.10}$$

where m labels the Kaluza-Klein modes and n the winding modes, which are both integer.

On the other hand, the stress-energy tensor of the field ϕ_1 is improved due to the background charge, and it is

$$T_1(z) = -\frac{1}{2}(\partial\phi_1)^2(z) + \frac{Q}{2}\partial^2\phi_1(z) \, . \tag{3.11}$$

The corresponding primary fields are vertex operators of the general form $V(z) = \exp(q\phi_1(z))$ with conformal dimensions equal to

$$h = -\frac{1}{2}q(q+Q) \tag{3.12}$$

with respect to $T_1(z)$. Choosing appropriate values for the momenta and the background charge of the Liouville field ϕ_1, and fixing the periodicity of the field ϕ_2 as follows,

$$q = l\sqrt{\frac{2}{k-2}}\,, \qquad Q = \sqrt{\frac{2}{k-2}}\,, \qquad R = \sqrt{k}R_0\,, \tag{3.13}$$

where $R_0 = 1/\sqrt{2}$ is the "self-dual" radius, we find that the net spectrum is the same as in equation (3.5) above. Similarly, identifications are worked out in the anti-holomorphic sector of the model.

The total stress-energy tensor of the conformal field theory (3.8) is

$$T(z) = T_1(z) + T_2(z) = -\frac{1}{2}(\partial\phi_1)^2(z) - \frac{1}{2}(\partial\phi_2)^2(z) + \frac{1}{\sqrt{2(k-2)}}\partial^2\phi_1(z) \tag{3.14}$$

with Virasoro central charge

$$c = 2\frac{k+1}{k-2}\,, \tag{3.15}$$

which is the same as for the coset model $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$. Therefore, the fields ϕ_1 and ϕ_2 can be used to provide a free field realization of the gauged WZW model, although the conformal field theory (3.8) is not meant to be equivalent to $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$. Also, as we will see later in section 5, the expression (3.14) provides the stress-energy tensor of the black-hole coset in terms of free fields that arise in the parafermionic construction of the underlying $\mathrm{SL}(2,\mathbb{R})_k$ current algebra, and it comes as no surprise that the two spectra coincide.

This construction suggests the rescaling of the Virasoro generators which is required before taking the limit of the theory at critical level. The infinite contribution of the background charge is removed by considering

$$\tilde{T}(z) = \lim_{k\to 2}\left(\sqrt{2(k-2)}T(z)\right) = \partial^2\phi_1(z)\,, \tag{3.16}$$

which satisfies the simplified operator product expansion

$$\tilde{T}(z)\tilde{T}(w) \sim \frac{1}{(z-w)^4}\,, \tag{3.17}$$

up to an overall (irrelevant) constant. The rescaling removes the infinity from the central charge of the Virasoro algebra, but the price to pay is the contraction of the Virasoro algebra to an abelian structure generated by the derivative of the U(1) current $\partial\phi_1$. Thus, $\mathrm{SL}(2,\mathbb{R})_2/\,\mathrm{U}(1)$ is a singular conformal field theory, but otherwise it makes perfect sense as a quantum theory.

The rescaling (3.16) forces the constituent fields that appear in the free field realization of the $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$ coset model to decouple at critical level, since only ϕ_1 is contributing

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to $\tilde{T}(z)$. The decoupling of the Liouville field ϕ_1 is also reflected in the rescaled form of the conformal dimensions that follow by multiplying (3.7) with $\sqrt{k-2}$. The first term still grows large as $1/\sqrt{k-2}$, whereas the contribution of ϕ_2 goes to zero as $\sqrt{k-2}$. Disregarding the very heavy states associated to the Liouville field, we obtain a tensionless model where gravity plays no role, but there is still some non-trivial structure associated to the field ϕ_2 that remains behind. It is our purpose to investigate some aspects of the residual theory in a more systematic way.

In addition, as we will see in section 5, the $\mathrm{SL}(2,\mathbb{R})_k$ current algebra admits a free field realization in terms of three scalar fields, using the parafermionic construction of the $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$ coset plus one extra boson. A degeneration takes place at $k=2$, since one of the two bosons that parametrize the parafermions of the coset decouples naturally from the rest. In this case, $\mathrm{SL}(2,\mathbb{R})_2$ makes perfect sense without need for rescaling of the affine currents, but the Sugawara construction is singular; it provides extra reason to believe that gravity plays no role in the tensionless limit. As for the remaining field, it assumes a non-geometric role in the exact description of the coset model. This unexpected reduction should also be held responsible for the null states that arise in the representation theory of the $\mathrm{SL}(2,\mathbb{R})_k$ algebra at $k=2$.

Finally, we also note for completeness that there is a rescaling of the Liouville field ϕ_1,

$$\tilde{\phi}(z) = \sqrt{2(k-2)}\,\phi_1(z)\,, \tag{3.18}$$

which can be consistently implemented in $\mathrm{SL}(2,\mathbb{R})_k$ and $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$ and leads to the following expression for the rescaled Virasoro operator

$$\tilde{T}'(z) = \lim_{k\to 2}\,(2(k-2)T(z)) = -\frac{1}{2}(\partial\tilde{\phi})^2(z) + \partial^2\tilde{\phi}(z) \tag{3.19}$$

at critical level. In this case, $\tilde{\phi}$ can be viewed as a null boson with

$$\langle\tilde{\phi}(z)\tilde{\phi}(w)\rangle = 0\,, \tag{3.20}$$

and as a result

$$\tilde{T}'(z)\tilde{T}'(w) \sim 0 \tag{3.21}$$

without having singular terms. Then, $\tilde{T}'(z)$ is strictly abelian, unlike (3.17), and it provides the central elements in the enveloping algebra of $\mathrm{SL}(2,\mathbb{R})_k$, which are non-trivial only for $k=2$. This rescaling is consistent but contrary to the previous case the resulting null field $\tilde{\phi}$ does not decouple from the operators of the WZW model at critical level. The systematic study of this limit also proves interesting in many respects as it connects quite naturally the Casimir (3.19) with the hamiltonian of completely integrable quantum spin chains, [42]. These issues will not be discussed here but in forthcoming publications.

3.3 Kac-Kazhdan determinant formula

We complete our general discussion by including the Kac-Kazhdan determinant formula for the highest weight representations of the $\mathrm{SL}(2,\mathbb{R})_k$ current algebra and the corresponding determinant of the inner product of states of the coset module $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$, [43]; see

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also [44] for a more comprehensive discussion of the subject. It offers a complementary understanding of the huge degeneracy that is expected to arise at critical level for the algebra as for the coset space model.

Let us consider the holomorphic sector of the current algebra and denote by D_N the determinant of inner products of all states in the highest weight module which are lying at a given $L_0^{\mathrm{SL}(2,\mathbb{R})}$ level N with J_0^3 charge equal to m; the anti-holomorphic sector will not be discussed, but it can be treated in a similar way. Clearly, the individual inner products of states depend on m and k, as well as the value of the Casimir operator $\mathcal{C}_2$. It is convenient to define the operators

$$\mathcal{J}_{+n} = \mathcal{C}_2 + (m+n-1)(m+n)\,, \qquad \mathcal{J}_{-n} = \mathcal{C}_2 + (m-n+1)(m-n) \tag{3.22}$$

for all $n > 0$. Then, for general values of k, the determinant assumes the form

$$\begin{aligned} D_N \, = \, & (-1)^{r_3(N)} C_N (k-2)^{r_3(N)} \prod_{n=1}^{N} \left(\mathcal{J}_{-n}\mathcal{J}_{+n}\right)^{P_3(N,n)} \times \\ & \times \prod_{r,s=1}^{N} \left(\mathcal{C}_2 + \frac{1}{4}\left(r(k-2)+s+1\right)\left(r(k-2)+s-1\right)\right)^{p_3(N-rs)} , \end{aligned} \tag{3.23}$$

where C_N is a positive numerical constant and r, s are constrained so that $rs \leq N$. The exponents are all positive but their exact form is not very important in the present work. Note that the determinant vanishes at $k = 2$, as noted before, due to the appearance of many null states.

The determinant formula for the representations of the coset model can also be found using the factorization

$$D_N = \prod_{q=0}^{N} D_N^{(q)} \,, \tag{3.24}$$

where $D_N^{(q)}$ denotes the determinant of inner products of states at level N of the $\mathrm{SL}(2,\mathbb{R})$ representation which are at level q of the U(1) current algebra. According to this rearrangement, $D_N^{(0)}$ provides the determinant of inner products of states in the coset space module, which is of interest here, and it assumes the following form

$$\begin{aligned} D_N^{(0)} \, = \, & C'_N \left(1-\frac{2}{k}\right)^{r_2(N)} \prod_{n=1}^{N} \left(\mathcal{J}_{-n}\mathcal{J}_{+n}\right)^{P_2(N,n)} \times \\ & \times \prod_{r,s=1}^{N} \left(C_2 + \frac{1}{4}\left(r(k-2)+s+1\right)\left(r(k-2)+s-1\right)\right)^{p_2(N-rs)} , \end{aligned} \tag{3.25}$$

where C'_N is another positive numerical constant, $rs \leq N$, and the exponents are appropriately chosen as before. Note that the factors appearing in the first line are all positive when $k > 2$ and $\mathcal{C}_2$ is taken in the continuous series representations, whereas the factors in the second line are positive when $k > 2$ and $\mathcal{C}_2 > 0$. Thus, the determinant of the coset module is always positive for the continuous series representations provided that $k > 2$, which is required for unitarity. In the tensionless limit we only have to consider continuous series representations, since the unitarity bound (3.6) squeezes all discrete representations, but the determinant also vanishes in this case.

In summary, for unitary highest weight representations all states have strictly positive norm and there are no non-trivial null states when $k > 2$. When the level of the $\mathrm{SL}(2,\mathbb{R})_k$ algebra becomes critical many null states make their appearance and lead to huge degeneracies of the spectrum.

4. Liouville field and null gauging

The background of a two-dimensional black-hole provides a classical solution of the same theory that describes a Liouville field coupled to $c = 1$ matter. The $c = 1$ matrix model superficially has a one-dimensional target space, as it appears in its initial formulation, but in fact it is more naturally understood in terms of a two-dimensional space, [45], where the extra dimension is provided by the Liouville mode. In view of this correspondence, the radial variable r of the coset model can be interpreted as Liouville field, which is always space-like in either euclidean or lorentzian version of the model. The relation becomes very clear in the weak coupling region $r \to \infty$, where the string coupling $\exp(-\Phi)$ tends to zero and the dilaton grows linearly as $\Phi \sim r$. Thus, in this region, the geometry of the black-hole is asymptotic to the two-dimensional geometry of the $c = 1$ matrix model, but their equivalence is not valid everywhere.

The precise relation between the two models is better understood by revisiting the interpretation of a (non-critical) string theory in $d - 1$ dimensions as a string theory in d dimensions with the extra dimension being provided by the Liouville field,[3] as in the $c = 1$ matrix model. Following [32], we note that the reverse map exists only if the gradient of the dilaton field of a d-dimensional model has the same space-time character with the Liouville coordinate; in this case, the extra coordinate can be gauged away, using conformal transformations, and one arrives at a lower dimensional model, as it can be easily seen from the modified transformation law of the target space variables $\delta X^\mu \sim G^{\mu\nu}\nabla_\nu\Phi$ to lowest order in α'. However, if the gradient of the dilaton changes character or if it becomes singular at certain points, the reverse map will break down. This is precisely the situation we encounter for the two-dimensional coset model and as a result *the black-hole cannot be regarded as a theory of $c = 1$ matter coupled to two-dimensional (Liouville) gravity.* This, in turn, suggests that the theory which remains after the decoupling of gravity in the tensionless limit of the coset model is not $c = 1$ matter in isolation; instead it is a variant of it or an exotic phase thereof, which is still poorly understood. We will briefly return to it in section 6, where the symmetries of the model will be represented as fermion bilinears and compared to the usual $W_{1+\infty}$ symmetry of the ordinary $c = 1$ matrix model.

[3] This relation is usually stated for critical strings in d dimensions, although here we want eventually to apply it to tensionless models where there is no notion of criticality; likewise, the conformal symmetry degenerates in the tensionless limit and cannot be employed in the argument. These issues introduce complications that invalidate the effective field theory description and they will not be addressed properly in the present context.

Next, in order to understand the role of the Liouville field in the tensionless limit, we consider the null gauging of WZW models. Equivalently, the same theory is obtained by making an infinite boost in the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$. In this case, we will find that the target space geometry exhibits a drastic reduction to only one dimension, which is always space-like and corresponds to the radial coordinate r. Then, since we are committed to interpreting the Liouville field as a spatial coordinate of the black-hole geometry and not as a (euclidean) time variable, it follows that the null gauging of $\mathrm{SL}(2,\mathbb{R})$ captures the gravitational sector of the model that decouples when $k=2$.

We proceed with some background material on the null gauging of WZW models for non-compact groups, putting the emphasis on $\mathrm{SL}(2,\mathbb{R})_k$; examples with higher rank groups will be presented later in section 8. It is well known that the group $\mathrm{SL}(2,\mathbb{R})$ has three different conjugacy classes of subgroups that are isomorphic either to the group of rigid rotations in two dimensions, $\mathrm{SO}(2)$, or the Lorentz group $\mathrm{SO}(1,1)$, or the isotropy group of light-like vectors, $E(1)$, which in this case coincides with the euclidean group in one dimension. Their generators are $i\sigma_2$, σ_3 and $\sigma^+ = \sigma_3 + i\sigma_2$, respectively, but in the latter case the generator is nilpotent since $(\sigma^+)^2 = 0$. The standard gauging of the $\mathrm{SL}(2,\mathbb{R})$ WZW model is taken either with respect to the group $\mathrm{SO}(2)$ or $\mathrm{SO}(1,1)$, thus giving rise to the euclidean or lorentzian two-dimensional black-hole cosets. The null gauging was also considered in the literature before, [46, 47, 48], by forming the $\mathrm{SL}(2,\mathbb{R})_k/E(1)$ WZW model, and a striking result was found, at least at it appears at first sight, namely that the classical target space geometry degenerates to one dimension. The result follows easily by applying the usual prescription for axial or vector gauging of the corresponding subgroup $E(1)$.

Let us consider the action of WZW models and discuss first their axial gauging. Using the standard parametrization of the $\mathrm{SL}(2,\mathbb{R})$ group elements g in the fundamental representation,

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix} ; \qquad ab + uv = 1 \,, \tag{4.1}$$

we fix the gauge by choosing $a + b = 0$. The parameters

$$\chi = u - v \,, \qquad w = a - b - u - v \tag{4.2}$$

remain invariant under axial transformations and provide a gauge invariant parametrization of the coset space $\mathrm{SL}(2,\mathbb{R})/E(1)$. Then, the action of the gauged WZW model assumes the form

$$S(g; A, \bar{A}) = \frac{k}{2\pi} \int d^2 z \left(\frac{\partial w \bar{\partial} w}{w^2} - \frac{1}{w^2} \mid A + \frac{1}{2w^2} (\chi \partial w - w \partial \chi) \mid^2 \right) , \tag{4.3}$$

which after the integration of the gauge fields yields the following metric and dilaton fields

$$ds^2 = k \frac{\partial w \bar{\partial} w}{2w^2} \,, \qquad \Phi = 2 \log w \tag{4.4}$$

in the large k limit. Finally, introducing a scalar field ϕ, so that $w = \pm \exp \phi$, depending on the sign of w, one arrives at the one-dimensional theory in target space

$$S = \frac{k}{2\pi} \int d^2z \, (\partial\phi\bar{\partial}\phi) + \frac{1}{4\pi} \int d^2z \sqrt{g} R\phi \, , \tag{4.5}$$

which includes the contribution of a linear dilaton and describes the action of a free scalar field with background charge. This is the theory of a Liouville field with zero cosmological constant, whereas the other field χ decouples from the geometry and it does not appear in the action.

We also note for completeness that the vector gauging proceeds in a similar way by making the gauge choice $u - v = 0$ and then using the two gauge invariant parameters w and $a + b$ to describe the coset. By eliminating the gauge fields, the same reduction occurs in target space, as before, and the classical geometry of the vector gauged $\mathrm{SL}(2,\mathbb{R})_k/E(1)$ model is described again by the effective action (4.5). Thus, the end result is insensitive to the gauging prescription, which appears to be self-dual with respect to T-duality transformations, and there is a drastic dimensional reduction in target space.

The same result admits an interesting group theoretic description by boosting the subgroup H that appears in ordinary gauged models to very large Lorentz factor, as an alternative to null gauging. For the simplest class of $\mathrm{SL}(2,\mathbb{R})_k$ models, let us consider the transformation

$$\begin{aligned} \sigma_1(\beta) &= e^{-\beta\sigma_1}\sigma_1 e^{\beta\sigma_1} = \sigma_1 \, , \\ i\sigma_2(\beta) &= e^{-\beta\sigma_1} i\sigma_2 e^{\beta\sigma_1} = (\sinh 2\beta)\sigma_3 + (\cosh 2\beta) i\sigma_2 \, , \\ \sigma_3(\beta) &= e^{-\beta\sigma_1}\sigma_3 e^{\beta\sigma_1} = (\cosh 2\beta)\sigma_3 + (\sinh 2\beta) i\sigma_2 \, , \end{aligned} \tag{4.6}$$

which introduces a boost with parameter β,

$$\tanh 2\beta = \frac{v}{c} \, . \tag{4.7}$$

It describes a Lorentz transformation in the Lie algebra of $\mathrm{SL}(2,\mathbb{R}) \simeq \mathrm{SO}(2,1)$ when there are two frames moving perpendicular to the spatial direction σ_1 with relative velocity v. As β ranges from 0 to infinity, $i\sigma_2(\beta)$ interpolates smoothly between $i\sigma_2(0) = i\sigma_2$ and $i\sigma_2(\infty)$, which becomes proportional to the nilpotent element σ^+. Likewise, $\sigma_3(\beta)$ interpolates between σ_3 and σ^+ as β ranges from 0 to infinity. Thus, one may use the group generated by $i\sigma_2(\beta)$ to gauge the $\mathrm{SL}(2,\mathbb{R})_k$ WZW model and obtain the geometry of $\mathrm{SL}(2,\mathbb{R})_k/E(1)$ in the infinite boost limit of the usual black-hole coset model. This prescription refers to the euclidean black-hole coset by boosting the compact subgroup generated by $i\sigma_2$, whereas the lorentzian model can be treated similarly provided that one boosts the generator of the non-compact subgroup $\sigma_3(\beta)$. In either case, one arrives at the same description of the coset model $\mathrm{SL}(2,\mathbb{R})_k/E(1)$ in terms of the Liouville field (4.5) which is always *space-like* and independent of the axial or vector gauging.

The gauging of the action can be worked out in detail for all values of β in order to obtain a systematic expansion of the target space geometry in powers of $\exp(-4\beta)$ for the boosted abelian subgroup. More precisely, using

$$i\sigma_2(\beta) = \frac{1}{2} e^{2\beta} \left(\sigma^+ - e^{-4\beta}\sigma^- \right) , \tag{4.8}$$

the action for the boosted model is written in the form

$$\begin{aligned} S(g; A, \bar{A}) = \; & S_{\rm WZW} - \frac{k}{2\pi} \int d^2z \, {\rm Tr} \left(A\sigma^+ \bar{\partial} g g^{-1} \pm \bar{A} \sigma^+ g^{-1} \partial g \pm A\bar{A} \sigma^+ g \sigma^+ g^{-1} \right) + \\ & + \frac{k}{2\pi} e^{-4\beta} \int d^2z \, {\rm Tr} \left(A\sigma^- \bar{\partial} g g^{-1} \pm \bar{A} \sigma^- g^{-1} \partial g \pm \right. \\ & \left. \qquad \pm A\bar{A} (\sigma^- g \sigma^+ g^{-1} + \sigma^+ g \sigma^- g^{-1} \pm 2) \right) \mp \\ & \mp \frac{k}{2\pi} e^{-8\beta} \int d^2z \, {\rm Tr} \left(A\bar{A} \sigma^- g \sigma^- g^{-1} \right) , \end{aligned} \tag{4.9}$$

using the nilpotent elements $\sigma^+ = \sigma_3 + i\sigma_2$ and $\sigma^- = \sigma_3 - i\sigma_2$. Here, the $\pm$ signs refer to the axial and vector gauging, respectively, which are both treated together. Also, A and $\bar{A}$ are rescaled appropriately to absorb the factor $\exp 2\beta$, which appears in the definition of the boosted generator and becomes infinite when $\beta \to \infty$. Fixing the gauge and performing the integration over the fields A and $\bar{A}$, the effective action turns out to be (4.5) plus subleading terms of order $\mathcal{O}(\exp(-4\beta))$ that account for the coupling of the other field χ. Further details can be found in the literature, [46, 47, 48].

The action that results in this case describes the classical geometry of the coset $\mathrm{SL}(2,\mathbb{R})_k/E(1)$ when k tends to infinity. Quantum corrections will also change k to $k-2$, as it is customary in WZW models, but there are no other modifications since the geometry of target space is one-dimensional. Rescaling ϕ with $\sqrt{(k-2)/2}$, so that the corrected kinetic term of Liouville theory (4.5) becomes canonically normalized, the linear dilaton term gives rise to the stress-energy tensor (3.11) with background charge $Q = \sqrt{2/(k-2)}$ that becomes infinite as k approaches 2. Thus, null gauging selects only the Liouville sector of the black-hole coset and provides the effective description of the radial coordinate r associated to the generator σ_1. On the other hand, the tensionless limit of the ordinary gauged WZW model exhibits a non-trivial structure after the decoupling of Liouville field, but its form appears to be non-geometric. The algebraic structure of the residual $\mathrm{SL}(2,\mathbb{R})_2/\mathrm{U}(1)$ model will be examined later using the world-sheet formulation of two-dimensional cosets.

We conclude this section with a few remarks that will further clarify the meaning of the infinite boost in the Lie algebra of non-compact WZW models from different points of view. We first consider the ungauged non-compact WZW model and explain how null strings arise by accelerating its semi-classical solutions to very high Lorentz factor. Then, the quantization of classical null strings can be attempted as usual, [2, 3], but their theory cannot capture the properties of the $\mathrm{SL}(2,\mathbb{R})_2$ model which is taken directly at the quantum level. We will also consider the gauged WZW model $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ and compare different limits that describe high energy strings.

Recall that the $\mathrm{SL}(2,\mathbb{R})_k$ WZW model admits short and long string solutions using the spectral flow of different geodesics in AdS_3, [39, 41]. Short strings correspond to *time-like geodesics* which are described by group elements

$$g = U \begin{pmatrix} \cos(\alpha\tau) & \sin(\alpha\tau) \\ -\sin(\alpha\tau) & \cos(\alpha\tau) \end{pmatrix} V \tag{4.10}$$

where U, V belong in $\mathrm{SL}(2,\mathbb{R})$ and τ is the world-sheet time coordinate; if $U = V = 1$, the solution represents a particle sitting at the center of AdS_3. Their monodromy matrix is

$$M = \begin{pmatrix} \cos(\alpha\pi) & \sin(\alpha\pi) \\ -\sin(\alpha\pi) & \cos(\alpha\pi) \end{pmatrix} \in \mathrm{SO}(2) \tag{4.11}$$

and belongs to the elliptic conjugacy class of $\mathrm{SL}(2,\mathbb{R})$ generated by $i\sigma_2$.

On the other hand, long strings correspond to *space-like geodesics* which are described by group elements

$$g = U \begin{pmatrix} e^{\alpha\tau} & 0 \\ 0 & e^{-\alpha\tau} \end{pmatrix} V \tag{4.12}$$

where U, V belong in $\mathrm{SL}(2,\mathbb{R})$; if $U = V = 1$, the solution is a straight line cutting the space-like section $t \equiv \tilde{\theta} = 0$ of AdS_3 diagonally. In this case, their monodromy matrix is

$$M = \begin{pmatrix} e^{\alpha\pi} & 0 \\ 0 & e^{-\alpha\pi} \end{pmatrix} \in \mathrm{SO}(1,1) \tag{4.13}$$

and belongs to the hyperbolic conjugacy class of $\mathrm{SL}(2,\mathbb{R})$ generated by σ_3.

Since both elements $i\sigma_2$ and σ_3 are transformed to σ^+ by performing a boost (4.6) with very high Lorentz factor, we find that the monodromy matrices (4.11) and (4.13) correspond to the parabolic conjugacy class $E(1)$ of $\mathrm{SL}(2,\mathbb{R})$ which arises by contraction in either case. As a result, there is no distinction between short and long strings since they both correspond to *null geodesics* in AdS_3 when $\beta \to \infty$. Thus, null strings make their appearance as spectral flows of geodesics in the $\mathrm{SL}(2,\mathbb{R})$ model in the limit that all string solutions are speeding with infinitely large Lorentz factor.[4] It should be noted in this context that the spectral flow stretches a geodesic solution in the time direction (parametrized by $\tilde{\theta}$) by adding $w\tau$ and rotates it around the center $r = 0$ of AdS_3 by adding $w\sigma$ to the angular coordinate θ. The resulting classical solutions depend on the spatial world-sheet coordinate σ and describe circular strings that wind w times around the center of the space. For null strings, however, the world-sheet is degenerate and every point moves independently along a null geodesic, [1, 7], as the speed of light on the world-sheet becomes effectively zero. This will be possible only if the corresponding semi-classical solutions have $w = 0$ so that there is no σ dependence. Put differently, the spectral flow is not a symmetry of the $\mathrm{SL}(2,\mathbb{R})_k$ model when the generators of the Lie algebra (4.6) are infinitely boosted, which also explains why the Hilbert space of WZW model does not contain states other than the long and short strings associated to the spectral flow of the continuous and discrete representations. In any case, we will not attempt to discuss the quantization of classical null strings, as the approach we are following here is taken directly at the quantum level of WZW models and the results are expected to be different: the current algebra $\mathrm{SL}(2,\mathbb{R})_2$ still exhibits the symmetry of spectral flow in the quantum tensionless limit and the world-sheet never becomes degenerate in our approach.

[4] As such they should be compared to the ultra-relativistic limit of rotating strings in $AdS_3 \times S^3$, which become effectively tensionless in the limit of large angular momentum in S^3, [49], and have applications to weakly coupled Yang-Mills theory within the AdS/CFT correspondence.

Next, passing to the coset model $\mathrm{SL}(2,\mathbb{R})_k / \mathrm{U}(1)$, we observe that only the spatial direction σ_1 survives when $\beta \to \infty$, since it is perpendicular to the direction of the boost and it remains unaffected. Thus, after all, it is not surprising that the null gauging yields a Liouville field for the radial coordinate r, which parametrizes the gravitational sector of the model. This reduction is quite different from the simplifications that arise in the vicinity of cosmological singularities, where the spatial points become causally disconnected and gravity reduces to one-dimensional system that depends only on time. Such a limit was studied extensively in four-dimensional gravity, following the original work of Belinski, Khalatnikov and Lifshitz (BKL), [50], but it also received attention in modern day string theory and in the small tension expansion of M-theory, [51]; alternatively, it can be viewed as a strong coupling limit of gravity so that the Planck mass becomes zero. In the BKL case, the bulk tensionless limit of the gravitational theories is defined in an ultra-relativistic way by letting the velocity of light in target space become zero. Then, time derivatives of fields dominate over their spatial gradients and they subsequently lead to an ultra-local reduction. As a result, the target space is parametrized by only one coordinate, which is clearly time-like for space-times with lorentzian signature; the same limit can also arise in spaces with euclidean signature, upon analytic continuation, but the variable that remains cannot be interpreted as the radial coordinate r.

It is instructive to study the difference between the two reductions in the context of high energy strings. Recall that there are two different ways to obtain a string of very high energy: either accelerate an ordinary microscopic string to very high Lorentz factor or stretch it to become very large of macroscopic size. However, not both of them correspond to the tensionless limit. The first case yields ultra-relativistic strings which are always tensionless as in the BKL limit. The tensionless limit also describes high energy strings which propagate in spaces with compact spatial directions because their size is bounded from above; as a result, the kinetic energy dominates the contribution of their tension and the strings become effectively tensionless in the high energy limit. However, for string propagation in non-compact spaces, the contribution of their tension can also grow large without bound and turn to macroscopic objects at very high energies. Clearly, in this case, the high energy limit is not ultra-relativistic and the strings do not behave as tensionless objects.

With this in mind, we may now look at string configurations in the semi-classical geometry of a two-dimensional black-hole (or AdS_3 for the same matter). The radial variable r parametrizes the non-compact spatial direction and high energy strings can also stretch to macroscopic size. There is no other way to describe tensionless high energy strings in these models but decouple the radial degree of freedom from the remaining variables. This provides a classical prescription that prevents the appearance of macroscopic high energy strings, but need not be always appropriate to use for addressing questions in quantum WZW models at critical level. However, it helps to explain the physical difference between the two dimensional reductions — BKL versus null gauging — and provides an intuitive (yet classical) way to think about the decoupling of gravity in the tensionless limit of WZW models. Of course, this decoupling also arises in the tensionless limit of the quantum theory, since otherwise the conformal dimensions (3.7) receive contributions from the Liouville field that become infinite at $k = 2$.

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5. World-sheet symmetries of the $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$ coset

In this section we outline the construction of an infinite dimensional chiral algebra, which acts as extended world-sheet symmetry of the gauged WZW model $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$ for all $k \geq 2$. The results we present here are based on earlier work using the concept of non-compact parafermions and their free field realization in terms of two scalar fields with background charge. We arrive at a rather complicated algebraic structure that simplifies considerably in two special limits, as $k \to \infty$ and $k \to 2$. We first review the essential details of our working framework, using the parafermion currents to define the W-algebra of coset models, and then apply the results to the tensionless limit where it is found that the world-sheet symmetry is a higher spin truncation of the algebra W_∞ with spin $s \geq 3$. In this case, the Virasoro algebra decouples consistently from all commutation relations, and contracts to an abelian structure. In the other limit, $k \to \infty$, the symmetry algebra of the model is W_∞, which is generated by all integer spins $s \geq 2$. Thus, the interpolation between the two algebraic structures may be used further to provide a systematic algebraic framework for studying the role of $1/\alpha'$ corrections in gauged WZW models.

5.1 Parafermions and W-algebras

The two-dimensional conformal field theories of gauged WZW models contain a collection of chirally conserved currents $\psi_l(z)$, and their hermitean conjugates $\psi_l^\dagger(z) = \psi_{-l}(z)$, which are semi-local fields that interpolate between bosons and fermions. In particular, the parafermion currents $\psi_l(z)$, and their anti-holomorphic partners $\bar{\psi}_l(\bar{z})$, have fractional conformal dimensions which are determined by the mutual locality exponent with respect to the monodromy properties of their correlation functions. Parafermion algebras were first introduced in the simplest family of $\mathrm{SU}(2)_N/\,\mathrm{U}(1)$ models, [29], which have a Z_N symmetry that accounts for their charges; here $l = 0, 1, \ldots, N-1$ and $\psi_0(z) = \psi_0^\dagger(z) = 1$. In this case, the central charge of the Virasoro algebra is $c_\psi = 2(N-1)/(N+2)$ and ranges from $1/2$ to 2, as N assumes all integer values $2, 3, \ldots$ which are allowed by unitarity. Their introduction proves advantageous for the systematic description of all primary fields on group manifolds and their cosets, such as $\mathrm{SU}(2)_N$ and $\mathrm{SU}(2)_N/\,\mathrm{U}(1)$, since the two classes of conformal field theories are related to each other by subtracting and then adding back free bosons. At the same time, parafermions can also be used to construct unitary representations of the $\mathcal{N} = 2$ superconformal algebra with central charge $c = 3N/(N+2) = c_\psi + 1$.

The basic structure of the parafermion algebra is described by the operator product expansions, [29],

$$\psi_{l_1}(z)\psi_{l_2}(w) = C_{l_1,l_2}(z-w)^{\Delta_{l_1+l_2}-\Delta_{l_1}-\Delta_{l_2}}\left(\psi_{l_1+l_2}(w) + \mathcal{O}(z-w)\right), \tag{5.1}$$

$$\psi_{l_1}(z)\psi_{l_2}^\dagger(w) = C_{l_1,-l_2-k}(z-w)^{-2\Delta_{l_2}}\left(\psi_{l_1-l_2}(w) + \mathcal{O}(z-w)\right), \tag{5.2}$$

$$\psi_l(z)\psi_l^\dagger(w) = (z-w)^{-2\Delta_l}\left(1 + \frac{2\Delta_l}{c_\psi}(z-w)^2 T_\psi(w) + \mathcal{O}(z-w)^3\right) \tag{5.3}$$

where C_{l_1,l_2} are appropriately chosen structure constants determined by associativity. Also, Δ_l is the conformal dimension of $\psi_l(z)$ and $\psi_l^\dagger(z)$, which equals to $l(N-l)/N$ in the simplest case of the $\mathrm{SU}(2)_N/\,\mathrm{U}(1)$ WZW model. It is also implicitly assumed that $l_1 > l_2$ in the operator product expansion (5.2). The operator product expansion (5.3), $\psi_l(z)\psi_l^\dagger(w)$, gives rise to the stress-energy tensor T_ψ with central charge c_ψ, as well as to a collection of other chiral currents with integer spin that appear to higher orders in the power series expansion. These currents, in turn, form the extended conformal algebra of the model, which is known as W-algebra; for example, the W-algebra of the coset $\mathrm{SU}(2)_N/\,\mathrm{U}(1)$ is generated by chiral fields with integer spin $2, 3, \ldots, N$ and it is denoted by W_N. Further generalizations to higher dimensional coset models are also known, but their structure is more intricate for non-abelian gauging. The commutation relations of W-algebras are in general non-linear. For a collection of papers on the subject, see, for instance, [52].

Here, we are mainly concerned with the generalization of parafermions to non-compact groups, such as $\mathrm{SL}(2,\mathbb{R})_k$, and the construction of the W-algebra of the corresponding coset model $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$ for all $k \geq 2$; special emphasis will be placed later in the particular limit $k = 2$. Recall that non-compact parafermions were initially introduced as a tool to break the $c = 3$ barrier of the $\mathcal{N} = 2$ superconformal algebra and construct unitary representations with $c > 3$ from the $\mathrm{SL}(2,\mathbb{R})_k$ algebra by subtracting and then adding back a free boson, [44, 53, 54]. We may formally pass from the compact to the non-compact coset by letting $N = -k$ with k ranging continuously from 2 to infinity. In this case, the number of independent parafermion fields $\psi_l(z)$ is infinite, as there is no Z_N symmetry to truncate the number of their components to $N-1$. Also, the operator product expansions (5.1)–(5.3) can be extended to the non-compact model in a straightforward way, setting

$$\Delta_l = \frac{l(k+l)}{k} \tag{5.4}$$

and

$$c_\psi = 2\frac{k+1}{k-2}\,. \tag{5.5}$$

These expressions follow from the corresponding values of the $\mathrm{SU}(2)_N/\,\mathrm{U}(1)$ coset model by changing N to $-k$, and likewise for the structure constants of the parafermion algebra, which are determined to be

$$C_{l_1,l_2} = \left(\frac{\Gamma(k+l_1+l_2)\Gamma(k)\Gamma(l_1+l_2+1)}{\Gamma(l_1+1)\Gamma(l_2+1)\Gamma(k+l_1)\Gamma(k+l_2)}\right)^{1/2}\,. \tag{5.6}$$

We also note that the operator product expansion of non-compact parafermion currents $\psi_{l_1}(z)\psi_{l_2}(w)$, which is shown in equation (5.1), contains no singular terms, since the exponents $\Delta_{l_1+l_2} - \Delta_{l_1} - \Delta_{l_2} = 2l_1l_2/k$ are always positive; thus, the operators $\psi_l(z)$ commute among themselves for all positive values of $l = 0, 1, 2, 3, \ldots$ Non-trivial commutation relations arise only among $\psi_l(z)$ and their hermitean conjugate partners.

Clearly, the infinitely generated algebra of non-compact parafermions allows for $c_\psi \geq 2$, and as a result the tensionless limit $c_\psi \to \infty$ is reached by letting $k \to 2$; this possibility does not arise for the compact coset model, since the central charge of its Virasoro algebra can never exceed the $c_\psi = 2$ barrier. The non-compact parafermion currents appear to have integer or half-integer dimensions in the tensionless limit, which are given by the special values

$$\Delta_l(k=2) = \frac{1}{2}l(l+2)\,; \qquad l = 0, 1, 2, 3, \ldots \tag{5.7}$$

In reality, however, these conformal dimensions are all zero since the Virasoro algebra with infinite central charge contracts to an abelian structure by appropriate rescaling of the stress-energy tensor; it is a simple consequence of the singular nature of the conformal field theory $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ at $k=2$. In any case, experience with operator algebras suggests that the corresponding W-algebra should be linear. In fact, we will be able to determine its exact algebraic structure and show that it can be identified with a consistent higher spin truncation of W_∞, whereas the Virasoro algebra becomes abelian and it decouples naturally from the spectrum of the world-sheet currents. This decoupling can already be seen in equation (5.3), since the coefficient of the term T_ψ becomes zero as $c_\psi \to \infty$; the same result holds after rescaling T_ψ by $\sqrt{k-2}$ in order to make sense of the tensionless limit as singular conformal field theory.

It is convenient to introduce free field realizations of the operators that arise in two-dimensional conformal field theories in order to simplify calculations with abstract operator algebras and compute correlation functions. Thus, for the $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ model, we introduce two free scalar fields $\{\phi_i(z); i=1,2\}$ and represent the parafermion currents $\psi_1(z)$ and $\psi_{-1}(z)$ as follows, [44, 53],

$$\psi_{\pm 1}(z) = \frac{1}{\sqrt{2k}}\left(\mp\sqrt{k-2}\,\partial\phi_1(z) + i\sqrt{k}\,\partial\phi_2(z)\right)\exp\left(\pm i\sqrt{\frac{2}{k}}\phi_2(z)\right) \tag{5.8}$$

for all $k \geq 2$. The fields $\phi_i(z)$ are both space-like with two point functions

$$\langle \phi_i(z)\phi_j(w)\rangle = -\delta_{ij}\log(z-w)\,. \tag{5.9}$$

The expressions (5.8) follow from the realization of $\mathrm{SU}(2)_N/\mathrm{U}(1)$ parafermion currents in terms of two free scalar fields, using the formal continuation $N \to -k$. The higher parafermion currents $\psi_l(z)$ also admit free field realizations, but their exact description will not be needed in the present work.

The parafermion algebra can be converted into the $\mathrm{SL}(2,\mathbb{R})_k$ current algebra by introducing operators

$$J^{\pm}(z) = \sqrt{k}\,\psi_{\pm 1}(z)\exp\left(\pm\sqrt{\frac{2}{k}}\chi(z)\right), \qquad J^3(z) = -\sqrt{\frac{k}{2}}\,\partial\chi(z)\,, \tag{5.10}$$

which dress the parafermions with the addition of an extra free scalar field $\chi(z)$; likewise, there is a parafermionic construction of the $\mathcal{N}=2$ superconformal algebra, [44]. We have, in particular, the operator product expansions[5]

$$\begin{aligned} J^+(z)J^-(w) &= \frac{k}{(z-w)^2} - 2\frac{J^3(w)}{z-w}\,, \\ J^3(z)J^{\pm}(w) &= \pm\frac{J^{\pm}(w)}{z-w}\,, \\ J^3(z)J^3(w) &= -\frac{k/2}{(z-w)^2}\,, \end{aligned} \tag{5.11}$$

[5]There are two different ways in the literature to define the currents $J^\pm$ by considering $J^1 \pm iJ^2$ or $iJ^1 \mp J^2$. Their hermiticity properties are different and likewise the hermiticity properties of the corresponding parafermions $\psi_{\pm1}$ are also different.

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which provide a free field realization of the $\mathrm{SL}(2,\mathbb{R})_k$ current algebra in terms of three scalar fields $\phi_1(z)$, $\phi_2(z)$ and $\chi(z)$. In this case, the stress-energy tensor of the coset model $\mathrm{SL}(2,\mathbb{R})_k\,/\,\mathrm{U}(1)$, which arises in the operator product expansion $\psi_1(z)\psi_1^\dagger(w)$, is realized in terms of two free scalar fields, as

$$W_2(z) \equiv T_\psi(z) = -\frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 + \frac{1}{\sqrt{2(k-2)}}\partial^2\phi_1\,, \tag{5.12}$$

where one of them, denoted by $\phi_1(z)$, appears with background charge and accounts for the value (5.5) of the central charge of the Virasoro algebra for all k. Note that the expression (5.12) coincides with the stress-energy tensor (3.14) that was introduced earlier in our discussion of the spectrum.

Following the bootstrap method of conformal field theory, we define the *primary* higher spin generators $W_s(z)$ of the extended conformal operator algebra of the model $\mathrm{SL}(2,\mathbb{R})_k\,/\,\mathrm{U}(1)$, using the expansion, [55],

$$\begin{aligned}\psi_1(z+\epsilon)\psi_{-1}(z) \,=\, \epsilon^{-2\frac{k+1}{k}}\Bigg(1 &+ \frac{k-2}{k}\left(\epsilon^2 + \frac{1}{2}\epsilon^3\partial + \frac{3}{20}\epsilon^4\partial^2 + \frac{1}{30}\epsilon^5\partial^3\right)W_2(z) - \\ &- \frac{1}{4}\left(\epsilon^3 + \frac{1}{2}\epsilon^4\partial + \frac{1}{7}\epsilon^5\partial^2\right)W_3(z) + \\ &+ \frac{(6k+5)(k-2)^2}{2k^2(16k-17)}\left(\epsilon^4 + \frac{1}{2}\epsilon^5\partial\right) : W_2^2 : (z) + \\ &+ \frac{1}{32}\left(\epsilon^4 + \frac{1}{2}\epsilon^5\partial\right)W_4(z) - \frac{(10k+7)(k-2)}{4k(64k-107)}\epsilon^5 : W_2W_3 : (z) - \\ &- \frac{1}{3\cdot 2^7}\epsilon^5 W_5(z) + \mathcal{O}(\epsilon^6)\Bigg)\,.\end{aligned} \tag{5.13}$$

Here, normal ordered products are defined, as usual, by subtracting the singular terms plus the finite terms that are total derivatives of lower dimension operators appearing in the operator product expansion. Then, using the free field realization of the parafermion currents $\psi_{\pm1}(z)$, we can obtain explicit expressions for the higher spin generators in terms of two free fields for all values of k. Of course, the case $k = 2$, which is relevant in the tensionless limit, is special because the parafermion currents (5.8) are realized in terms of one scalar field only, ϕ_2, and likewise for the resulting higher spin generators $W_s(z)$; this case will be treated separately later.

We may extract the higher spin generators and compute their operator product expansions in order to identify the structure of the resulting W-algebra for all values of k. We will follow the general construction presented in reference [55]. The spin 3 operator turns out to be

$$\begin{aligned}W_3(z) \,=\, 2i\sqrt{\frac{2}{k}}\Bigg(&\frac{3k-4}{3k}(\partial\phi_2)^3 + \frac{1}{6}\partial^3\phi_2 + \frac{k-2}{k}(\partial\phi_1)^2\partial\phi_2 + \\ &+ \frac{k-2}{k}\sqrt{\frac{k-2}{2}}\partial^2\phi_1\partial\phi_2 - \sqrt{\frac{k-2}{2}}\partial\phi_1\partial^2\phi_2\Bigg),\end{aligned} \tag{5.14}$$

which yields the following operator product expansion with itself

$$
\begin{aligned}
W_3(z+\epsilon)W_3(z) \;=\; & \frac{16}{3}\frac{(k+1)(k+2)(3k-4)}{k^3}\frac{1}{\epsilon^6} + 16\frac{(k+2)(k-2)(3k-4)}{k^3} \times \\
& \times\left(\frac{1}{\epsilon^4}+\frac{1}{2}\frac{\partial}{\epsilon^3}+\frac{3}{20}\frac{\partial^2}{\epsilon^2}+\frac{1}{30}\frac{\partial^3}{\epsilon}\right)W_2(z) + 2\frac{2k-3}{k}\left(\frac{1}{\epsilon^2}+\frac{1}{2}\frac{\partial}{\epsilon}\right)W_4(z) + \\
& + 2^7\frac{(k+2)(3k-4)(k-2)^2}{k^3(16k-17)}\left(\frac{1}{\epsilon^2}+\frac{1}{2}\frac{\partial}{\epsilon}\right) : W_2^2 : (z)\,. \qquad (5.15)
\end{aligned}
$$

Here, $W_4(z)$ is the primary spin 4 operator which is defined by the parafermionic operator product expansion (5.13), and it turns out to be

$$
\begin{aligned}
W_4(z) \;=\; & -\frac{4(3k-4)}{k^2(16k-17)}\left((k-12)(2k-1)(\partial\phi_2)^4 - 2(2k^2+2k+3)(\partial^2\phi_2)^2\right) - \\
& -\frac{4(k-2)}{k^2(16k-17)}\Big(8(k^2-k+1)\left(\partial\phi_2\partial^3\phi_2+\partial\phi_1\partial^3\phi_1\right) + (k-2)(6k+5)(\partial\phi_1)^4 + \\
& \qquad + 6(2k^2-13k+8)(\partial\phi_1)^2(\partial\phi_2)^2 - 2(6k^2-12k+1)(\partial^2\phi_1)^2\Big) + \\
& +\frac{8\sqrt{2(k-2)}}{3k^2(16k-17)}\Big(3(k-2)(6k+5)(\partial^2\phi_1)(\partial\phi_1)^2 - 6k(16k-17)\partial\phi_1\partial\phi_2\partial^2\phi_2 + \\
& \qquad + 3(2k-3)(19k-8)\partial^2\phi_1(\partial\phi_2)^2 + (k^2-k+1)\partial^4\phi_1\Big). \qquad (5.16)
\end{aligned}
$$

Higher spin generators and their operator product expansions can be constructed recursively in a similar fashion, using the free field realization of the parafermions $\psi_{\pm 1}(z)$, but their structure becomes quickly rather involved for all $2 < k < \infty$.

For generic values of the Kac-Moody level k, it is very difficult to iterate the algorithm and extract the structure of the underlying W-algebra in closed form together with the free field realization of its higher spin generators. However, the boundary values $k = 2$ and $k = \infty$ are rather special, since many simplifications occur and the whole procedure becomes tractable. As we will see later, there are special *quasi-primary* bases of the W-algebra that remove all non-linear terms in a systematic way when k reaches its boundary values. This result is also closely related to the local character of the parafermion currents at $k = 2$ and $k = \infty$, since $\psi_{\pm 1}(z)$ are represented as derivatives of fermions and a boson, respectively. The details will be made available shortly.

Before we proceed further, let us recall a number of qualitative results which are known for arbitrary k by performing sample calculations with generators up to spin 5, [55]. First, the extended conformal symmetry of the $\mathrm{SL}(2,\mathbb{R})_k\,/\,\mathrm{U}(1)$ coset model appears to be infinitely generated[6] by chiral operators $W_s(z)$ with all integer spin $s \geq 2$. Second, for

[6]It is natural to expect that the extended conformal symmetries of non-compact coset models are infinitely generated as they correspond to irrational conformal field theories. In these cases there are no finitely generated W-algebras whose representations can be used to organize the operator content in terms of a finite number of primary field blocks, unlike the simpler case of compact coset models. Likewise, the family of non-compact parafermions $\psi_{\pm l}(z)$ is infinitely large, in contrast to the compact space parafermions, which form a finite family. It should be noted, however, that the W-algebra that arises here has null fields of spin $s \geq 6$, and as a result it appears to be non-freely generated for all values of k, [56]. This occurrence is closely connected to the existence of non-trivial unitary quasi-free representations of W_∞-type algebras, but we will not consider their implications any further.

generic values of k, the resulting W-algebra is non-linear as there is no field redefinition that can bring it into a linear form. Thus, it is a non-linear deformation of the W_∞ algebra, with coefficients that depend on k; for this reason, the world-sheet symmetry of the black-hole coset is denoted by $\hat{W}_\infty(k)$ for generic k. Third, we may formally extend the validity of the algebra $\hat{W}_\infty(k)$ to all real values $-\infty < k < +\infty$ and set $k = -N$, where N is any positive integer greater or equal than 2. Then, it appears that all higher spin generators with $s > N$ become null, as it can be seen by considering the two-point functions $\langle W_s(z+\epsilon)W_s(z)\rangle$. In this case, all generators with $s > N$ can be consistently set equal to zero, thus rendering $\hat{W}_\infty(k=-N)$ finitely generated and isomorphic to the algebra W_N. This result is also consistent with the formal relation between the two coset models $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$ and $\mathrm{SU}(2)_N/\,\mathrm{U}(1)$ for $k = -N$. Thus, the algebra $\hat{W}_\infty(k)$ is rather universal, as it can be viewed as a continuous generalization of the W_N algebras for all *real* values of the level k.

Within this general framework we also encounter the world-sheet symmetry of the tensionless non-compact coset model $\mathrm{SL}(2,\mathbb{R})_2/\,\mathrm{U}(1)$, which is clearly identified with $\hat{W}_\infty(k=2)$. The detailed description of its algebraic structure is one of the primary goals of the present work. At the same time, the non-linear algebra $\hat{W}_\infty(k)$, which appears for generic values of k, provides a concrete framework for exploring the symmetries of the model at finite tension, and may offer an understanding of the exact nature of $1/\alpha'$ corrections from the world-sheet viewpoint; it is an important problem with far reaching consequences that should be investigated in the future. Finally, we note for completeness that much larger symmetries may arise in the tensionless limit with $\hat{W}_\infty(k=2)$ being the smallest subalgebra of a much larger world-sheet symmetry group; for example, we may include all higher parafermion currents $\psi_{\pm l}(z)$ and investigate whether they all form an enlarged symmetry group together with the higher spin generators $W_s(z)$ at $k=2$. Such generalizations are also lying beyond the scope of the present work.

The complete structure of the $\hat{W}_\infty(k)$ algebra can be determined alternatively, without relying on the specific realization of the parafermion currents in terms of free fields, using the correlation functions of the elementary fields $\psi_{\pm 1}(z)$. Recall at this point the recursive relations among the parafermionic correlation functions,

$$\begin{aligned}
&\langle \psi_1(z_1)\cdots\psi_1(z_n)\psi_1^\dagger(w_1)\cdots\psi_1^\dagger(w_n)\rangle = \\
&\quad = \prod_{i=2}^{n}(z_1-z_i)^{2/k}\prod_{j=1}^{n}\frac{1}{(z_1-w_j)^{2/k}}\times \\
&\qquad \times\sum_{a=1}^{n}\left(\frac{1}{(z_1-w_a)^2}-\frac{2}{k(z_1-w_a)}\left(\sum_{l=2}^{n}\frac{1}{w_a-z_l}-\sum_{m\neq a}\frac{1}{w_a-w_m}\right)\right)\times \\
&\qquad \times\prod_{q=2}^{n}\frac{1}{(z_q-w_a)^{2/k}}\prod_{p=1}^{a-1}(w_p-w_a)^{2/k}\prod_{r=a+1}^{n}(w_a-w_r)^{2/k}\times \\
&\qquad \times\langle\psi_1(z_2)\cdots\psi_1(z_n)\psi_1^\dagger(w_1)\cdots\hat{\psi}_1^\dagger(w_a)\cdots\psi_1^\dagger(w_n)\rangle\,,
\end{aligned} \tag{5.17}$$

which determine all such $2n$-correlation functions in terms of lower $2(n-2)$-correlators. They follow from the corresponding relations for the compact coset parafermions, [29], by setting $k = -N$. Then, since $\langle\psi_1(z)\psi_1^\dagger(w)\rangle = 1/(z-w)^{2(k+1)/k}$, as follows from equation (5.3), we easily obtain

$$\langle\psi_1(z_1)\psi_1^\dagger(z_2)\psi_1(z_3)\psi_1^\dagger(z_4)\rangle = \left(\frac{z_{13}z_{24}}{z_{12}z_{14}z_{34}z_{23}}\right)^{2/k}\left(\frac{1}{z_{12}^2 z_{34}^2}\left(1-\frac{2}{k}\frac{z_{12}z_{34}}{z_{23}z_{24}}\right)+(z_2\leftrightarrow z_4)\right), \tag{5.18}$$

where $z_{ij} = z_i - z_j$. Likewise, we may obtain explicit expressions for the six-point correlation functions and so on.

Using the parafermionic four-point correlation function (5.18), as well as the operator product expansion (5.13) that contains the chiral fields $W_s(z)$ in power series, we may obtain the two-point correlation functions among the W-algebra generators

$$\begin{aligned}
\langle W_2(z)W_2(0)\rangle &= \frac{k+1}{k-2}\frac{1}{z^4}\,,\\
\langle W_3(z)W_3(0)\rangle &= \frac{16}{3}\frac{(k+1)(k+2)(3k-4)}{k^3}\frac{1}{z^6}\,,\\
\langle W_4(z)W_4(0)\rangle &= \frac{2^{10}(k+1)(k+2)(k+3)(2k-1)(3k-4)}{k^4(16k-17)}\frac{1}{z^8}\,,\\
\langle W_5(z)W_5(0)\rangle &= \frac{9\cdot 2^{15}(k+1)(k+2)(k+3)(k+4)(2k-1)(5k-8)}{5k^5(64k-107)}\frac{1}{z^{10}}\,,
\end{aligned} \tag{5.19}$$

and so on for all other higher spin fields. They account for the central terms appearing in the commutation relations of the non-linear algebra $\hat{W}_\infty(k)$. Continuing further, we may compute the structure constants of the algebra that appear as coefficients in the singular terms of the operator product expansion $W_s(z+\epsilon)W_{s'}(z)$ for all generators. This is achieved by considering the parafermionic six-point function $\langle\psi_1(z_1)\psi_1^\dagger(z_2)\psi_1(z_3)\psi_1^\dagger(z_4)\psi_1(z_5)\psi_1^\dagger(z_6)\rangle$, which is naturally related to the three-point functions of W-generators when combined with the expansion (5.13). Unfortunately, although this procedure is straightforward, it is rather cumbersome to implement in all generality in order to extract the complete structure of $\hat{W}_\infty(k)$ in closed form for arbitrary values of the level k. Thus, either way, the exact structure of the non-linear algebra $\hat{W}_\infty(k)$ remains out of reach in all generality.

The remaining part of this section is devoted to the derivation of the linear structures that arise in the two special limits $k\to\infty$ and $k\to 2$. The first is already known in the literature and it will be only briefly discussed for completeness. The second is new and it appears here for the first time. Both limiting values will be given a systematic description in section 6 in the framework of W_∞-type algebras, [57, 58, 59, 60, 61].

5.2 W-algebra at $k=\infty$

The algebra $\hat{W}_\infty(k)$ linearizes in the limit $k\to\infty$, as it can be easily seen by introducing an appropriate *quasi-primary* basis for its higher spin generators. Note that in the large k limit the basic parafermion fields become local and they simplify to the U(1) currents

$$\psi_1(z) = i\partial\phi(z)\,,\qquad \psi_{-1}(z) = i\partial\bar{\phi}(z)\,, \tag{5.20}$$

where $\phi(z)$ is a complex free boson

$$\phi(z) = \frac{1}{\sqrt{2}}\partial(\phi_2 + i\phi_1)\,. \tag{5.21}$$

Then, their operator product expansion can be written in the form

$$\psi_1(z+\epsilon)\psi_{-1}(z) = \frac{1}{\epsilon^2}\left(1+\sum_{s=2}^{\infty}\frac{(-1)^s(2s-1)!\epsilon^s}{2^{2(s-2)}(s-1)!(s-2)!}\sum_{n=0}^{\infty}\frac{(s+n-1)!\epsilon^n}{n!(2s+n-1)!}\partial^n\tilde{W}_s(z)\right) \tag{5.22}$$

where $\tilde{W}_s(z)$ are appropriately chosen *quasi-primary* generators that absorb all non-linear terms of the operator product expansion (5.13) as $k\to\infty$.

The W-generators that result in this case are given by simple bilinear expressions in free field realization

$$\begin{aligned}\tilde{W}_2(z) &= W_2(z) = -\partial\phi\partial\bar{\phi}\,,\\ \tilde{W}_3(z) &= W_3(z) = -2\left(\partial\phi\partial^2\bar{\phi}-\partial^2\phi\partial\bar{\phi}\right),\\ \tilde{W}_4(z) &= W_4(z)+6:W_2^2:(z) = -\frac{16}{5}\left(\partial\phi\partial^3\bar{\phi}-3\partial^2\phi\partial^2\bar{\phi}+\partial^3\phi\partial\bar{\phi}\right)\end{aligned} \tag{5.23}$$

and so on. $\tilde{W}_2(z)$ is the stress-energy tensor of the model with central charge equal to its classical value $c=2$, whereas all other operators $\tilde{W}_s(z)$ are higher spin currents with $s=3,4,\ldots$

More generally, it is known that the complete system of *quasi-primary* operators is given in the large k limit by the general expression, [58],

$$\tilde{W}_s(z) = \frac{2^{s-3}s!}{(2s-3)!!(s-1)}\sum_{k=1}^{s-1}(-1)^k\binom{s-1}{k}\binom{s-1}{k-1}\partial^k\phi\partial^{s-k}\bar{\phi} \tag{5.24}$$

for all values of spin $s\geq 2$. Obviously, the operator product expansion among these operators, $\tilde{W}_s(z)\tilde{W}_{s'}(w)$, leads to a linear algebra thanks to the bilinear form of the generators (5.24). The commutation relations can also be written in Fourier modes using the standard prescription

$$[\tilde{W}_n^s,\tilde{W}_m^{s'}] = \oint_{C_0}\frac{dw}{2\pi i}w^{m+s'-1}\oint_{C_w}\frac{dz}{2\pi i}z^{n+s-1}\tilde{W}_s(z)\tilde{W}_{s'}(w)\,, \tag{5.25}$$

where C_w is a contour around w and C_0 a contour around 0. The algebra that results in this case is denoted by W_∞ and its structure will be described in the next section together with several other technical details that are also relevant for the tensionless limit of the coset model.

5.3 W-algebra at $k=2$

Next, we examine the structure of $\hat{W}_\infty(k)$ when the level of the $\mathrm{SL}(2,\mathbb{R})_k$ algebra assumes its critical value, $k=2$. In this case, the background charge of the ϕ_1 boson becomes infinite and appropriate rescaling of the stress-energy tensor is required in order to make the parafermionic Virasoro algebra well defined. We first rescale the operator $W_2(z)$ shown in equation (5.12) by $\sqrt{k-2}$ and then take the limit $k\to 2$, which leads to the identification $\tilde{W}_2(z)\sim\partial^2\phi_1$, as in equation (3.16) before. The rescaling amounts to an abelian contraction of the Virasoro algebra, since $\tilde{W}_2(z)$ is the derivative of a U(1) current with $\tilde{W}_2(z+\epsilon)\tilde{W}_2(z) \sim 1/\epsilon^4$. Introducing Fourier modes

$$\tilde{L}_n = \frac{1}{2\pi i}\oint_0 dz \tilde{W}_2(z) z^{n+1} \times \begin{cases} 1/n\,, & \text{for } n \neq 0\,, \\ 1\,, & \text{for } n = 0\,, \end{cases} \tag{5.26}$$

which are also conveniently rescaled by n, we arrive at the U(1) current algebra

$$[\tilde{L}_n, \tilde{L}_m] = n\delta_{n+m,0} \tag{5.27}$$

that replaces the Virasoro algebra of the coset model at $k = 2$. It is realized by the field ϕ_1 alone.

On the other hand, the parafermion currents simplify at $k = 2$ and they assume the following form

$$\psi_{\pm 1}(z) = \frac{i}{\sqrt{2}}\partial\phi_2(z) e^{\pm i\phi_2(z)}\,. \tag{5.28}$$

These currents depend only on the scalar field $\phi_2(z)$ and they are well defined without the need for rescaling. Then, their operator product expansion gives rise to W-generators, as usual, but they do not contain the stress-energy tensor in the spectrum. The decoupling arises from the independence of $\psi_{\pm 1}(z)$ from $\phi_1(z)$, whereas $\tilde{W}_2(z)$ depends only on it. It can also be seen directly from the operator product expansion of the parafermions (5.13), where the rescaling of $W_2(z)$ by $\sqrt{k-2}$ eliminates all W_2-dependent terms when $k \to 2$. Then, in this limit, we claim that the operator product expansion (5.13) can be written in a simpler form, which is analogous to the expansion (5.22),

$$\psi_1(z+\epsilon)\psi_{-1}(z) = \frac{1}{\epsilon^3}\left(1 + \sum_{s=3}^{\infty} \frac{(-1)^s(2s-1)!\epsilon^s}{2^{2(s-2)}(s-1)!(s-2)!} \sum_{n=0}^{\infty} \frac{(s+n-1)!\epsilon^n}{n!(2s+n-1)!}\partial^n \tilde{W}_s(z)\right)\,, \tag{5.29}$$

using a new system of appropriately chosen generators $\tilde{W}_s(z)$ for all $s \geq 3$. This basis is constructed by absorbing all composite W-operators that arise in the operator product expansion (5.13), as for the large k limit.

It is important to realize in this context that the parafermion currents $\psi_{\pm 1}(z)$ become total derivatives of a more elementary system of free fermions with components $(\Psi, \bar{\Psi})$, which are defined to be

$$\Psi(z) = e^{-i\phi_2(z)}\,, \qquad \bar{\Psi}(z) = e^{i\phi_2(z)} \tag{5.30}$$

with two-point function

$$\langle \Psi(z)\bar{\Psi}(w)\rangle = \frac{1}{z-w} = \langle \bar{\Psi}(z)\Psi(w)\rangle\,. \tag{5.31}$$

In reality they form the components of a fermionic ghost and conjugate ghost system with dimension 1/2. Then, for $k=2$, the operator product expansion of the conjugate fields

$$\psi_1(z) = \frac{1}{\sqrt{2}}\partial\bar{\Psi}(z)\,, \qquad \psi_{-1}(z) = -\frac{1}{\sqrt{2}}\partial\Psi(z) \tag{5.32}$$

can only give rise to fermion bilinears, in close analogy with the boson bilinears that arise in the operator product expansion of the parafermions for $k\to\infty$. As a result, it is also expected here that the operator product expansion of the new currents $\tilde{W}_s(z)$ will lead to an infinite dimensional algebra with linear commutation relations. Its structure will be subsequently determined by extracting the exact form of $\tilde{W}_s(z)$ as fermion bilinears and computing their operator product expansions using the two-point function (5.31). Equivalently, we may first bosonize the complex fermions using a free scalar boson ϕ_2, as given by the defining relations (5.30), and express all formulae in terms of ϕ_2.

We summarize below the result of the computations that were performed in this case. We have the following realization of the W-generators as fermion bilinears

$$\tilde{W}_s(z) = \frac{2^{s-4}(s+1)!}{(2s-3)!!(s-1)}\sum_{k=0}^{s-3}(-1)^k\binom{s-1}{k}\binom{s-1}{k+2}\partial^{k+1}\bar{\Psi}\partial^{s-k-2}\Psi(z) \tag{5.33}$$

for all $s\geq 3$, which are obtained by taking into account the normalizations appearing in equation (5.29) above. The lowest lying field is $\tilde{W}_3(z) = 2\partial\bar{\Psi}\partial\Psi(z)$, which equals to the normal ordered product $-4:\psi_1\psi_{-1}:(z)$ as required by the power series expansion (5.29). Next, we have $\tilde{W}_4(z) = 8(\partial\bar{\Psi}\partial^2\Psi - \partial^2\bar{\Psi}\partial\Psi)$ and so on for all other higher currents that can be readily found from equation (5.33). The result is analogous to the free field realization (5.24) that was encountered in the limit $k\to\infty$, but it is not the same.

The currents can be equivalently written using the free field $\phi_2(z)$. Their bosonization leads to the following expressions for the W-generators,

$$\begin{aligned}
\tilde{W}^3(z) &= \frac{i}{3}\left(2(\partial\phi_2)^3 + \partial^3\phi_2\right), \\
\tilde{W}^4(z) &= 4\left((\partial\phi_2)^4 + (\partial^2\phi_2)^2\right), \\
\tilde{W}^5(z) &= \frac{8i}{35}\left(-84(\partial\phi_2)^5 + 90(\partial\phi_2)^2\partial^3\phi_2 - 240\partial\phi_2(\partial^2\phi_2)^2 + \partial^5\phi_2\right), \\
\tilde{W}^6(z) &= \frac{64}{3}\left(-4(\partial\phi_2)^6 + 12(\partial\phi_2)^3\partial^3\phi_2 - 24(\partial\phi_2)^2(\partial^2\phi_2)^2 - 2(\partial^3\phi_2)^2 + 2(\partial^2\phi_2)\partial^4\phi_2\right), \\
\tilde{W}^7(z) &= \frac{128i}{693}\Big(1980(\partial\phi_2)^7 - 11970(\partial\phi_2)^4\partial^3\phi_2 + 21420(\partial\phi_2)^3(\partial^2\phi_2)^2 + 420(\partial\phi_2)^2\partial^5\phi_2 + \\
&\quad + 6300\partial\phi_2(\partial^3\phi_2)^2 - 6720\partial\phi_2\partial^2\phi_2\partial^4\phi_2 + 630(\partial^2\phi_2)^2\partial^3\phi_2 + \partial^7\phi_2\Big),
\end{aligned} \tag{5.34}$$

and so on. The first representatives of this list should be compared with the corresponding expressions for the higher spin fields derived for general values of k after taking the limit of the currents (5.14) and (5.16) at critical level, and they turn out to be the same. The higher spin fields $\tilde{W}_s(z)$ provide the right basis for the algebra by absorbing the non-linear terms that arise at higher spins. It is rather difficult to extract the free boson realization of all currents in closed form, but this is not really a handicap of our general construction as we already have them expressed as fermion bilinears for all $s\geq 3$.

It is now straightforward procedure to verify that the operator products of these currents take the form

$$
\begin{aligned}
\tilde{W}^3(z+\epsilon)\tilde{W}^3(z) &= \frac{16}{\epsilon^6} + \left(\frac{1}{\epsilon^2} + \frac{1}{2}\frac{\partial}{\epsilon}\right)\tilde{W}^4(z)\,, \\
\tilde{W}^3(z+\epsilon)\tilde{W}^4(z) &= 96\left(\frac{1}{\epsilon^4} + \frac{1}{3}\frac{\partial}{\epsilon^3} + \frac{1}{14}\frac{\partial^2}{\epsilon^2} + \frac{1}{84}\frac{\partial^3}{\epsilon}\right)\tilde{W}^3(z) + \frac{5}{3}\left(\frac{1}{\epsilon^2} + \frac{2}{5}\frac{\partial}{\epsilon}\right)\tilde{W}^5(z)\,, \\
\tilde{W}^3(z+\epsilon)\tilde{W}^5(z) &= \frac{1440}{7}\left(\frac{1}{\epsilon^4} + \frac{1}{4}\frac{\partial}{\epsilon^3} + \frac{1}{24}\frac{\partial^2}{\epsilon^2} + \frac{1}{180}\frac{\partial^3}{\epsilon}\right)\tilde{W}^4(z) + \frac{9}{4}\left(\frac{1}{\epsilon^2} + \frac{1}{3}\frac{\partial}{\epsilon}\right)\tilde{W}^6(z)\,, \\
\tilde{W}^4(z+\epsilon)\tilde{W}^4(z) &= \frac{1536}{\epsilon^8} + 8\left(\frac{36}{\epsilon^4} + 18\frac{\partial}{\epsilon^3} + 5\frac{\partial^2}{\epsilon^2} + \frac{\partial^3}{\epsilon}\right)\tilde{W}^4(z) + 3\left(\frac{1}{\epsilon^2} + \frac{1}{2}\frac{\partial}{\epsilon}\right)\tilde{W}^6(z)\,, \\
\tilde{W}^4(z+\epsilon)\tilde{W}^5(z) &= \frac{138240}{7}\left(\frac{1}{\epsilon^6} + \frac{1}{3}\frac{\partial}{\epsilon^5} + \frac{1}{14}\frac{\partial^2}{\epsilon^4} + \frac{1}{84}\frac{\partial^3}{\epsilon^3} + \frac{5}{3024}\frac{\partial^4}{\epsilon^2} + \frac{1}{5040}\frac{\partial^5}{\epsilon}\right)\tilde{W}^3(z) + \\
&\quad + \frac{4080}{7}\left(\frac{1}{\epsilon^4} + \frac{2}{5}\frac{\partial}{\epsilon^3} + \frac{1}{11}\frac{\partial^2}{\epsilon^2} + \frac{1}{66}\frac{\partial^3}{\epsilon}\right)\tilde{W}^5(z) + \frac{3}{5}\left(\frac{7}{\epsilon^2} + 3\frac{\partial}{\epsilon}\right)\tilde{W}^7(z)\,, \\
\tilde{W}^3(z+\epsilon)\tilde{W}^6(z) &= \frac{10240}{\epsilon^6}\tilde{W}^3(z) + \frac{1120}{3}\left(\frac{1}{\epsilon^4} + \frac{1}{5}\frac{\partial}{\epsilon^3} + \frac{3}{110}\frac{\partial^2}{\epsilon^2} + \frac{1}{330}\frac{\partial^3}{\epsilon}\right)\tilde{W}^5(z) + \\
&\quad + \frac{2}{5}\left(\frac{7}{\epsilon^2} + 2\frac{\partial}{\epsilon}\right)\tilde{W}^7(z)\,, \qquad (5.35)
\end{aligned}
$$

and so on for higher spin commutators.

The infinite dimensional algebra $\hat{W}_\infty(2)$ that result in tensionless limit of the non-compact coset $\mathrm{SL}(2,\mathbb{R})_k\,/\,\mathrm{U}(1)$ can be systematically described as a higher spin truncation of $W_{1+\infty}$. As we will see next, its form is rather unique and provides the extended symmetry algebra of the model in closed form. The same framework also helps to describe the linear form of $\hat{W}_\infty(k)$ that arises in the large k limit in a unifying way. For all other values, $2 < k < \infty$, the complete structure of $\hat{W}_\infty(k)$ still remains out of reach because of the non-linear terms in the commutation relations.

6. $W_{1+\infty}$ and its higher spin truncations

We set up the framework by considering the infinite dimensional algebra of all differential operators on the circle, namely $\{f(x)D^n; n = 0,1,2,\ldots\}$, where D denotes the derivative operator with respect to $x \in S^1$. Their algebra assumes the form

$$[f(x)D^n, g(x)D^m] = \big(nf(x)g'(x) - mf'(x)g(x)\big)\,D^{n+m-1} + \text{lower order terms}\,, \qquad (6.1)$$

where the subleading terms are lower order differential operators that follow by making use of Leibnitz's rule

$$D^n f(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) D^{n-k}\,. \qquad (6.2)$$

The leading order terms in equation (6.1) give rise to the algebra of area preserving diffeomorphisms on the cylinder $T^\star S^1$, whereas the inclusion of all subleading terms provides a non-linear deformation of it, which is also known as Moyal algebra on $T^\star S^1$.

It is more convenient in the sequel to introduce a complex parameter $z = e^{ix}$ and work with Laurent series in z, instead of trigonometric functions of x defined on the circle, and use ∂ to denote the derivative operator with respect to z. Then, the differential operators $\{z^{n+s-1}\partial^{s-1}\}$ with $n \in Z$ and $s \in Z^+$ provide a basis for writing the commutation relations of the resulting infinite dimensional Lie algebra. It contains the centerless Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} , \tag{6.3}$$

which is generated by the first order differential operators $L_n = -z^{n+1}\partial$ with $s = 2$ and it is associated with the algebra of point canonical transformation on $T^\star S^1$. Then, the complete algebra of all differential operators can be viewed as a module of the centerless Virasoro algebra with each $z^{n+s-1}\partial^{s-1}$ term having conformal weight s, [57].

The infinite dimensional algebra $W_{1+\infty}$ is defined to be the central extension of the algebra of all differential operators on S^1. The central extensions are described systematically in the mathematics literature using the logarithm of the derivative operator, $\log \partial$, whose commutator is defined to be

$$[\log \partial, A] = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} a^{(k)}(z)\partial^{n-k} \tag{6.4}$$

when acting on a differential operator $A = a(z)\partial^n$. The result is a pseudo-differential operator that involves negative powers of the derivative operator, and therefore it is natural to consider the pairing

$$\mathcal{C}(A, B) = \int \mathrm{res}\left([A, \log \partial] \circ B\right) \tag{6.5}$$

among any two differential operators A and B, [61, 62, 63]. The computation is performed using the calculus of pseudo-differential operators and *res* is the residue function of the resulting operator $[A, \log \partial] \circ B$ given by the coefficient of its ∂^{-1} term. For example, for the case of first order differential operators, one finds

$$\mathcal{C}(z^{n+1}\partial, z^{m+1}\partial) = \frac{1}{6}(n^3 - n) \int z^{n+m-1} dz \sim (n^3 - n)\delta_{n+m,0} , \tag{6.6}$$

which coincides with the usual cocycle formula of the Virasoro algebra that describes central extensions of (6.3). More generally, it is known that the 2-cocycle (6.5) provides the unique non-trivial central extension of the algebra of all differential operators on the circle, [62, 63]. It can be computed explicitly for all elements of the algebra, since

$$\mathcal{C}(f(z)\partial^n, g(z)\partial^m) = \frac{n!m!}{(n+m+1)!} \int f^{(m)}(z) g^{(n+1)}(z) dz . \tag{6.7}$$

After this brief outline of the exact mathematical structure, it is convenient to write down the complete system of commutation relations of the algebra $W_{1+\infty}$ in closed form by introducing a basis that diagonalizes the resulting central terms. Using the system of differential operators, [61],

$$V_n^s = -B(s) \sum_{k=1}^{s} \alpha_k^s \binom{n+s-1}{k-1} z^{n+s-k} \partial^{s-k} , \tag{6.8}$$

which consist of special series of operators of order up to $s-1$ for all $s=1,2,3,\ldots$, with coefficients

$$B(s)=\frac{2^{s-3}(s-1)!}{(2s-3)!!}\,,\qquad \alpha_k^s=\frac{(2s-k-1)!}{[(s-k)!]^2}\,,\tag{6.9}$$

we find that the 2-cocycle assumes the form

$$\mathcal{C}(V_n^s,V_m^{s'})=-\frac{B^2(s)}{2s-1}\frac{(n+s-1)!}{(n-s)!}\delta_{s,s'}\delta_{n+m,0}\,,\tag{6.10}$$

and it is diagonal in the indices s and s'.

Furthermore, it can be shown that the commutation relations of the centrally extended algebra of differential operators assume the following form, [59],

$$\begin{aligned}[V_n^s,V_m^{s'}]=&\ \big((s'-1)n-(s-1)m\big)\,V_{n+m}^{s+s'-2}+\sum_{r\geq1}g_{2r}^{ss'}(n,m;\mu)V_{n+m}^{s+s'-2-2r}+\\ &+c_s(\mu)n(n^2-1)(n^2-4)\cdots(n^2-(s-1)^2)\delta_{s,s'}\delta_{n+m,0}\,,\end{aligned}\tag{6.11}$$

where the structure constants of the algebra are given by

$$g_{2r}^{ss'}(n,m;\mu)=\frac{\phi_{2r}^{ss'}(\mu)}{2(2r+1)!}\,N_{2r}^{ss'}(n,m)\,,\tag{6.12}$$

with

$$\phi_{2r}^{ss'}(\mu)=\sum_{k=0}^{r}\frac{(-\frac{1}{2}-2\mu)_k(\frac{3}{2}+2\mu)_k(-r-\frac{1}{2})_k(-r)_k}{k!(-s+\frac{3}{2})_k(-s'+\frac{3}{2})_k(s+s'-2r-\frac{3}{2})_k}\tag{6.13}$$

and

$$N_{2r}^{ss'}(n,m)=\sum_{k=0}^{2r+1}(-1)^k\binom{2r+1}{k}(2s-2r-2)_k[2s'-k-2]_{2r+1-k}[s-1+n]_{2r+1-k}[s'-1+m]_k\,.\tag{6.14}$$

In the above formulae $(a)_n$ and $[a]_n$ denote the ascending and descending Pochhammer symbols, respectively,

$$(a)_n=a(a+1)\cdots(a+n-1)\,,\qquad [a]_n=a(a-1)\cdots(a-n+1)\tag{6.15}$$

with $(a)_0=1=[a]_0$. Finally, the coefficients of the central terms are given by the expression

$$c_s(\mu)=\frac{c}{2^{1-2|\mu|}}\frac{2^{2(s-3)}(s+2\mu)!(s-2\mu-2)!}{(2s-1)!!(2s-3)!!}\,,\tag{6.16}$$

where the overall coefficient c is left arbitrary and coincides with the value of the central charge of the Virasoro subalgebra generated by the Fourier modes V_n^2.

The commutation relations of the full $W_{1+\infty}$ algebra correspond to the choice of the free parameter $\mu=-1/2$, in which case the s-dependent coefficient of the central terms (6.16) coincides with the expression derived from the 2-cocycle formulae (6.10) in the basis (6.8). Other choices of μ describe various consistent truncations of $W_{1+\infty}$ that we will encounter shortly. For $\mu = -1/2$, the generators V_n^s represent the Fourier modes parametrized by $n \in Z$ of chiral fields with spin s,

$$V^s(z) = \sum_{n=-\infty}^{+\infty} V_n^s z^{-n-s} \,, \tag{6.17}$$

in the framework of two-dimensional conformal field theories. Indeed, the commutation relations (6.11) of $W_{1+\infty}$ can be converted into operator product expansions

$$V^s(z)V^{s'}(w) \sim -\sum_{r\geq 0} f_{2r}^{ss'}(\partial_z, \partial_w; \mu) \frac{V^{s+s'-2-2r}(w)}{z-w} - c_s(\mu)\delta_{s,s'}\partial_z^{2s-1}\frac{1}{z-w} \tag{6.18}$$

with

$$f_{2r}^{ss'}(n,m;\mu) = \frac{\phi_{2r}^{ss'}(\mu)}{2(2r+1)!} M_{2r}^{ss'}(n,m) \,, \tag{6.19}$$

where

$$M_{2r}^{ss'}(n,m) = \sum_{k=0}^{2r+1} (-1)^k \binom{2r+1}{k} (2s-2-2r)_k [2s'-k-2]_{2r+1-k} n^{2r+1-k} m^k \,. \tag{6.20}$$

The operators $M_{2r}^{ss'}(\partial_z, \partial_w)$ are obtained by replacing the powers appearing in n and m by the corresponding derivative operators with respect to z and w, respectively.

Thus, in the framework of two-dimensional conformal field theories, $W_{1+\infty}$ can be regarded as an extended world-sheet symmetry generated by an abelian U(1) current $V^1(z)$, the stress-energy tensor $V^2(z)$ with central charge c, which is determined by the conformal field theory that realizes the symmetry, and an infinite collection of higher spin fields $V^s(z)$ for all integer values $s \geq 3$. For $\mu = -1/2$, we observe that the operator product expansion $V^s(z)V^{s'}(w)$ contains all fields with spin $s+s'-2r$, starting from $s+s'-2$ and terminating at $V^2(w)$ or $V^1(w)$ for $s+s'$ even or odd, respectively. In this case, $W_{1+\infty}$ is formulated as infinite dimensional linear algebra by choosing a *quasi-primary* field basis with $\langle V^s(z)V^{s'}(w)\rangle \sim \delta_{s,s'}/(z-w)^{s+s'}$. Note that in a *primary* basis, which requires the use of non-linear field redefinitions, the commutation relations of the algebra $W_{1+\infty}$ are non-linear and, hence, less tractable in closed form.

Higher spin truncations of $W_{1+\infty}$ cannot be obtained by simply setting some of the lower spin generators equal to zero, as this prescription is not consistent with the Lie algebra commutation relations. Instead, consistent truncations can be made systematic by twisting the generators

$$\tilde{V}_n^s = V_n^s + \text{lower spin terms} \tag{6.21}$$

so that the new elements $\tilde{V}_n^s$ close among themselves for all values of s bigger or equal than a fixed integer value M, which provides the lower spin bound. The truncation is consistent when the remaining lower spin generators do not appear in the commutation relations of the generators with $s \geq M$; of course, non-trivial commutation relations will arise among the lower and higher spin generators so that the complete structure of the $W_{1+\infty}$ algebra is only described in a different basis. For example, as it is well known, the infinite dimensional algebra W_∞ generated by all fields with spin $s \geq 2$ follows from $W_{1+\infty}$ by appropriate twisting. Likewise, the twisting procedure can be generalized to construct higher spin algebras with $s \geq M$ for any choice of the lower cutoff integer M. We will present their construction using appropriate choice of bases in the algebra of all differential operators, from which explicit twisting formulae can be obtained for the higher spin truncations of $W_{1+\infty}$. Such higher spin algebras can be studied as mathematical curiosities on their own right, but here we find that they also characterize the chiral algebra of the gauged WZW models in the tensionless limit. Their precise meaning will become evident shortly by elaborating on the relevance of the lower spin generators that decouple entirely from the spectrum. In this context, the twisting (6.21) is only a method for their systematic construction from $W_{1+\infty}$, while their physical interpretation depends on the circumstances that they arise in quantum field theory.

It can be shown that the following system of differential operators of order bigger or equal than $M-1$ and less or equal than $s-1$,

$$\tilde{V}_n^s = -\frac{(s-M)!}{(s-1)!} B(s) \sum_{k=1}^{s-M+1} \frac{(s-k)!}{(s-k-M+1)!} \alpha_k^s \binom{n+s-1}{k-1} z^{n+s-k} \partial^{s-k} \, , \tag{6.22}$$

where $B(s)$ and α_k^s are given by equation (6.9), as before, form a closed algebra for all integer values of spin $s \geq M$, [61]. The logarithmic 2-cocycle remains diagonal in this basis and the commutation relations of the corresponding centrally extended higher spin algebra assume the general form (6.11)-(6.14) with parameter

$$\mu = \frac{1}{2}(M-2) \, . \tag{6.23}$$

Also, for this choice of μ, the central terms are given by equation (6.16) for all s, up to an overall numerical value which is conveniently normalized to $c/2^{1-2|\mu|}$ and it depends on the model. Note that the operator product expansions $\tilde{V}^s(z)\tilde{V}^{s'}(w)$ that emerge in this case involve all terms of the form $\tilde{V}^{s+s'-2r}$ starting from $\tilde{V}^{s+s'-2}(w)$ and terminating at $\tilde{V}^{M+1}(w)$ or $\tilde{V}^M(w)$ for $s+s'+M-1$ even or odd, respectively. The resulting infinite dimensional algebra will be denoted by $W_\infty^{(M)}$ and involves all generators with $s \geq M$. If $M \geq 3$, the higher spin algebra is not conformal, as it does not contain the Virasoro algebra, and the term "spin" should not be taken at face value; nevertheless, we will continue to call higher spin algebras all such consistent truncations of the original $W_{1+\infty}$ algebra.

It is now straightforward to put the results of the previous section into a more systematic framework. First, the algebra $\hat{W}_\infty(k)$ linearizes in the large k limit and becomes isomorphic to the algebra W_∞ for which $M=2$. In this case the free field realization of the *quasi-primary* operators (5.24) that follow from the parafermionic operator product expansion satisfy the commutation relations (6.11) with $\mu = 0$. This result is already known in the literature, [58], and it will not be discussed further. Second, and most important result for the purposes of the present work is the identification of $\hat{W}_\infty(k)$ at critical level $k=2$ with the higher spin algebra $W_\infty^{(M)}$ for $M=3$. Indeed, it can be verified that the free field realization of the operators (5.33) satisfy the commutation relations (6.11) with

$\mu = 1/2$ provided that the following rescaling of generators is also taken into account,

$$\tilde{W}_n^s = \frac{s-2}{2}\tilde{V}_n^s \,. \tag{6.24}$$

The linear structure of the algebra $\hat{W}_\infty(2)$ was already established in the previous section for all generators, thanks to the bilinear form of the currents $\tilde{W}_s(z)$ in terms of the complex fermion system $(\Psi, \bar{\Psi})$. Also, the operator product expansions (5.35) can be identified with the corresponding commutation relations of $W_\infty^{(3)}$ after the rescaling (6.24). Then, these sample calculations suggest the exact equivalence of the algebras $\hat{W}_\infty(2)$ and $W_\infty^{(3)}$ for all values of $s \geq 3$. The complete proof relies on the uniqueness of the algebra $W_{1+\infty}$, and its higher spin truncations, following from the uniqueness of the Moyal algebra as linear deformation of the algebra of area preserving diffeomorphism of $T^\star S^1$, [64], and the uniqueness of its central extensions, [62, 63]. Besides, this can also be verified independently by direct computation of the commutation relations among all other higher spin fields, which thus prove their exact equivalence.

Summarizing, we have shown that *the chiral operator algebra of the gauged WZW model* $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ *at critical level* $k = 2$ *is the higher spin truncation of* $W_{1+\infty}$ *generated by chiral fields with* $s \geq 3$. In this case we find that the fermionic realization (5.33) yields central terms for the higher spin generators with coefficients $c_s(\mu = 1/2)$ given by equation (6.16) for

$$c = 1 \,. \tag{6.25}$$

The value of the central charge does not have the usual meaning as in conformal field theories because the chiral algebra $W_\infty^{(3)}$ does not contain the Virasoro algebra, which is decoupled from the spectrum. The Virasoro algebra contracts to an abelian structure which commutes with all other generators $\tilde{W}_s(z)$ after rescaling of the Virasoro generators by $\sqrt{k-2}$. Thus, in the present case, one should think of the algebra $W_\infty^{(3)}$ as having life on its own, and it cannot be extended to the full $W_{1+\infty}$ symmetry by (un)twisting the generators. Otherwise, the quantum field theory of the tensionless coset $\mathrm{SL}(2,\mathbb{R})_2/\mathrm{U}(1)$ would remain conformal after the decoupling of the Liouville field; this possibility is ruled out by the singular character of the Sugawara construction in terms of $\mathrm{SL}(2,\mathbb{R})$ currents as there is no stress-energy tensor in the spectrum at critical level.

It is also instructive to compare the free field representation (5.33) with the fermionic representation of $W_\infty^{(3)}$ that results by twisting the higher spin generators of $W_{1+\infty}$. Recall that $W_{1+\infty}$ also admits a free field realization with fermion bilinears

$$V_s(z) = 2^{s-3}\frac{(s-1)!}{(2s-3)!!}\sum_{k=0}^{s-1}(-1)^k \binom{s-1}{k}^2 \partial^k\bar{\Psi}\partial^{s-k-1}\Psi(z)\,, \tag{6.26}$$

which satisfy the commutation relations (6.11) with Virasoro central charge $c = 1$, [60]. Then, introducing appropriate field redefinitions $\tilde{V}_n^s = V_n^s +$ lower spin terms, we obtain new elements $\tilde{V}_n^s$ that represent the algebra of higher spin operators (6.22) as fermion bilinears, [61],

$$\tilde{V}_s(z) = 2^{s-3}\frac{(s-M)!(s+M-2)!}{(s-1)!(2s-3)!!}\sum_{k=0}^{s-M}(-1)^k\binom{s-1}{k}\binom{s-1}{k+M-1}\partial^k\bar{\Psi}\partial^{s-k-1}\Psi(z) \tag{6.27}$$

with $s \geq M$ and central terms having $c = (-1)^{M-1}2^{1-2|\mu|}$, according to our normalizations. The central charge is easily computed by noting that the lowest spin operator $\tilde{V}_M(z)$ and its two-point function are given by

$$\tilde{V}_M(z) = 2^{2(M-2)}\bar{\Psi}\partial^{M-1}\Psi(z)\,; \qquad \langle\tilde{V}_M(z)\tilde{V}_M(w)\rangle = (-1)^{M-1}\frac{[2^{2(M-2)}(M-1)!]^2}{(z-w)^{2M}}\,. \tag{6.28}$$

Then, passing to Fourier modes and comparing with the coefficient of the central term (6.16) for $s = M$ and $\mu = (M-2)/2$, we arrive at the value of c given above.

This procedure provides another fermionic realization of the algebra $W_\infty^{(M)}$ for all integer values of the lower spin M, which can be subsequently specialized to $M = 3$ and compared with the representation (5.33). The two realizations are different from each other since the corresponding spin 3 operators $\tilde{V}_3(z)$ are $4\bar{\Psi}\partial^2\Psi(z)$ and $4\partial\bar{\Psi}\partial\Psi(z)$, respectively. Likewise, their higher spin representatives are also different from each other, although both of them have $c = 1$ when $M = 3$. It should be noted, however, that they differ by total derivative terms of lower spin fields including $\bar{\Psi}\partial\Psi(z) - (\partial\bar{\Psi})\Psi(z)$ and $\bar{\Psi}\Psi(z)$. These two expressions provide the stress-energy tensor and the U(1) number current of the conformal field theory of free fermions, but they do not represent physical operators of the coset model at critical level. Put differently, $\mathrm{SL}(2,\mathbb{R})_2/\,\mathrm{U}(1)$ is not meant to be equivalent to the theory of free fermions, but only the algebra of its operators is realized in terms of some fermion bilinears; all other fermionic expressions, like $\bar{\Psi}\Psi$ and $\bar{\Psi}\partial\Psi - (\partial\bar{\Psi})\Psi$, do not correspond to operators of the coset model. Thus, it is not surprising that there is no physical Virasoro generator to append to the chiral algebra of the coset $\mathrm{SL}(2,\mathbb{R})_2/\,\mathrm{U}(1)$ and extend the free field realization (5.33) of $W_\infty^{(3)}$ to the full $W_{1+\infty}$ algebra. As we have already stressed, the coset model is a singular conformal field theory in the tensionless limit.

This comparison may also have important consequences for the characterization of the $\mathrm{SL}(2,\mathbb{R})_2/\,\mathrm{U}(1)$ quantum theory in connection with the $c = 1$ matrix model. It is well known that in the fermionic description of the matrix model there are infinitely many conserved quantities associated with conserved currents of higher spin of the form (6.26), which form a $W_{1+\infty}$ algebra with Virasoro central charge $c = 1$ (see, for instance, [65]); they include the fermion number associated with the spin-1 current $\bar{\Psi}\Psi$ followed by the stress-energy tensor and a collection of higher spin generators. One the other hand, it has already been noted in section 4 that the $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$ model can not be regarded as a theory of $c = 1$ matter coupled to two-dimensional gravity, since this identification is only valid in the asymptotic (weak coupling) region of the classical geometry. Therefore, in the tensionless limit, where gravity decouples in the form of a Liouville field, the remnants cannot be possibly identified with the ordinary $c = 1$ model; instead, the theory that remains should be thought as a variant or an exotic phase of the $c = 1$ matrix model. The fermionic realization of the corresponding symmetry algebra $W_\infty^{(3)}$ suggests that this unknown model could also be formulated in terms of fermion fields, as in the ordinary case, but without having lower spin currents among its physical operators. We expect that better understanding will be achieved in the future by studying the matrix model for the two-dimensional black-hole, [66], which is based on a conjectured equivalence between the $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ coset and the sine-Liouville model, and the dynamics of vortices.

7. BRST analysis of the world-sheet symmetry

In bosonic string theory, where the Virasoro algebra is the underlying world-sheet symmetry, nilpotency of the BRST operator

$$\mathcal{Q} = \sum_{n=-\infty}^{+\infty} L_n c_n - \frac{1}{2} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} (n-m) : c_{-n} c_{-m} b_{n+m} : , \tag{7.1}$$

provides the critical value of the central charge $c = 26$, [67]. Here, b_n and c_n are the Fourier modes of a fermionic ghost system (b, c) with conformal weights $(2, -1)$, respectively, which are associated with reparametrization invariance. On the other hand, when the Virasoro algebra contracts to a U(1) current algebra, as in the commutation relations $[L_n, L_m] = n\delta_{n+m,0}$ for the case of tensionless strings, the corresponding BRST operator squares to zero without the need to impose any restrictions on the coefficient of the central term, [4, 5, 6]. Thus, the concept of critical dimension, which renders the quantum theory of strings free of Weyl anomalies, appears to be lost when $\alpha' \to \infty$. It comes as no surprise, since the very notion of space-time where the evolution of strings takes place is not relevant in the tensionless limit, and the cancellation of the Weyl anomaly that otherwise breaks space-time Lorentz invariance is not an issue anymore. The result is also consistent with the arbitrariness of the coefficient that remains in the central terms after rescaling the Virasoro generators to absorb the infinite value of the central charge when $\alpha' \to \infty$. It is for this reason that $\mathrm{SL}(2,\mathbb{R})_2/\mathrm{U}(1)$ can be taken as model for tensionless strings, although it does not arise as a limiting case within critical string theory.

The purpose of this section is to show that for two-dimensional quantum field theories that possess additional higher spin symmetries, the BRST charge is not nilpotent unless the coefficient of the corresponding higher spin central terms is fixed to a critical value. Thus, although the contracted Virasoro symmetry is not sufficient to distinguish among different tensionless models, nilpotency of the BRST charge for the $W_\infty^{(3)}$ world-sheet symmetry may be used, instead, to impose severe restrictions on tensionless string model building. There is an implicit assumption that one makes in this case, namely that $W_\infty^{(3)}$ serves as a fundamental world-sheet symmetry in the tensionless limit. Then, cancellation of the anomalies associated to all higher spin generators with $s \geq 3$ will provide a substitute of the critical dimension, but there is no space-time interpretation of its value as gravity has decoupled. We will find that the central terms (6.16), which correspond to the choice $\mu = 1/2$,

$$c_s\left(\mu = \frac{1}{2}\right) = \frac{2^{2(s-3)}(s-3)!(s+1)!}{(2s-3)!!(2s-1)!!} c \tag{7.2}$$

cancel by the corresponding ghost contributions provided that the overall coefficient is fixed to the value

$$c = 2\,, \qquad \forall\ s \geq 3\,. \tag{7.3}$$

The analysis we perform in this section can only be viewed at the present time as curiosity of the symmetries arising in the tensionless limit of gauged WZW models. It should be used in a more fundamental way in case the theory of tensionless strings admits a systematic reformulation as W-strings for the non-conformal algebra $W_{\infty}^{(3)}$. Such point of view will not be confirmed here, but it will be investigated separately as a viable possibility in future work. Apart from this issue, it should be mentioned that the BRST analysis also serves as an additional consistency check of the truncation procedure leading to the infinite dimensional algebra $W_{\infty}^{(3)}$ and its non-trivial central extensions.

Recall that the BRST operator for a Lie algebra with generators T^a and structure constants $f^{ab}{}_c$ is generally given by

$$\mathcal{Q} = c_a T^a - \frac{1}{2} f^{ab}{}_c c_a c_b b^c \tag{7.4}$$

where (c_a, b^a) is a pair of Faddeev-Popov ghosts that possess opposite statistics to the generators T^a; as such, they satisfy $\{c_a, b^b\} = \delta_a^b$ and all other anticommutators vanish. For finite dimensional Lie algebras with trivial cohomology groups, as for all simple Lie algebras, the operator $\mathcal{Q}$ is always nilpotent and there is no anomaly that needs to be cancelled. However, for infinite dimensional algebras that admit non-trivial central extensions, the quantum theory may be anomalous as the ghost contributions do not always balance the bosonic part of the central charge in order to have $\mathcal{Q}^2 = 0$. This is common to all symmetries arising in two-dimensional conformal field theory, thus leading to critical values of the central charges.

The BRST operator of the Virasoro algebra, and all other algebras that contain higher spin fields among the generators, can be constructed by first introducing a system of ghost fields $(b_s(z), c_s(z))$ for each current $V_s(z)$ with weights $(s, 1-s)$, respectively, and

$$\langle b_s(z) c_s(w) \rangle = \frac{1}{z-w} = \langle c_s(z) b_s(w) \rangle\,. \tag{7.5}$$

Then, the operator $\mathcal{Q}$ is defined to be the charge

$$\mathcal{Q} = \frac{1}{2\pi i} \oint dz j(z) \tag{7.6}$$

associated to the BRST current of the corresponding symmetry algebra

$$j(z) = \sum_{\{s_i\}} V_{s_i}(z) c_{s_i}(z) + \cdots\,. \tag{7.7}$$

Here, the summation is taken over all generators of the extended world-sheet symmetry and dots denote appropriately chosen ccb-type terms whose exact form depends on the structure constants of the underlying Lie algebra; for instance, for the Virasoro algebra generated by the stress-energy tensor with spin 2, we have $j(z) = T(z)c_2(z) + c_2 \partial c_2 b_2(z)$, which yields the BRST operator (7.1) in terms of Fourier modes. One may also work out similar expressions for W-algebras, as we will see later in detail.

The nilpotency of the BRST operator, $\mathcal{Q}^2 = 0$, implies that the operator product expansion of the currents $j(z)j(w)$ vanishes up to total derivative terms. This is precisely how the Virasoro anomaly cancels in bosonic string theory to yield $c = 26$. For extended conformal symmetries, the ghost contribution to the central term of the Virasoro subalgebra comes from all possible higher spin generators and it turns out to be

$$c_{\rm gh}(2) = -\sum_{\{s_i\}}(6s_i^2 - 6s_i + 1)\,. \tag{7.8}$$

Each (b_s, c_s) system contributes the characteristic value $-(6s^2 - 6s + 1)$. Thus, for finitely generated W-algebras, the total contribution to $c_{\rm gh}(2)$ is finite and its value depends on the spin content of the additional symmetries. Anomaly cancellation requires $c/2 + c_{\rm gh}(2) = 0$, which fixes c to a given critical value. For instance, for Zamolodchikov's W_3 algebra, which is generated by the stress-energy tensor and an additional chiral field with spin 3, the ghost contribution is $-13 - 37 = -50$ and therefore the critical central charge of W_3-strings is $c = 100$, [68]. For infinitely generated algebras, as for $W_{1+\infty}$ and its higher spin truncations that were encountered before, the sum that determines $c_{\rm gh}(2)$ diverges and additional regularizations have to be taken into account in order to make good sense of it. Also, in all these cases, one has to make sure that the coefficients of the singular terms that arise in the operator product expansion $j(z)j(w)$ vanish all at once to all orders in $z - w$. The central terms of the higher spin generators receive different ghost contributions $c_{\rm gh}(s)$ that appear to order $1/(z-w)^{2s}$ for each value of s and their cancellation is a prerequisite for the consistency of any regularization scheme that makes the BRST operator nilpotent. This singles out a definite value for the central charge c for which the W-algebra becomes free of anomalies.

There is a definite regularization scheme that makes $c_{\rm gh}(2)$ finite for all infinitely generated conformal algebras, which can also be extended to all higher spin central charges $c_{\rm gh}(s)$ in a consistent way. This problem was investigated for the first time for the algebra W_∞, where the formal expression (7.8) can be made finite using the zeta function regularization,

$$c_{\rm gh}(2) = -\sum_{j=0}^{\infty}\big(6(j+1)^2 + 6(j+1) + 1\big) = -6\zeta(-2,1) - 6\zeta(-1,1) - \zeta(0,1)\,, \tag{7.9}$$

while setting for convenience $j = s - 2 \geq 0$. Here, $\zeta(s,a)$ is defined as usual

$$\zeta(s,a) = \sum_{n\geq 0}(n+a)^{-s}\,, \tag{7.10}$$

which converges for $s > 1$, but it has a pole at $s = 1$. Using the analytic continuation of (7.10) to negative integer values $s = -l$, we define the regularized zeta function in terms of the Bernoulli polynomials[7]

$$\zeta(-l,a)_{\rm reg} = -\frac{B_{l+1}(a)}{l+1}\,. \tag{7.11}$$

[7]We list the first few Bernoulli polynomials that are needed for the calculations presented here and in the remaining part of this section: $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$, $B_3(x) = x^3 - 3x^2/2 + x/2$,

Then, we easily find that the regularized value of the ghost contribution to the Virasoro algebra is $c_{\rm gh}(2) = 1$, [69]. Hence, the BRST operator of W_∞ is nilpotent at the spin 2 level provided that the bosonic representation of the algebra has the opposite central charge, $c = -2$. Furthermore, it can be verified that this choice of central charge is also critical for the higher spin generators of the algebra, thus leading to similar anomaly cancellations between $c_{\rm gh}(s)$ and the "matter" part of the corresponding higher order central terms. Likewise, for the $W_{1+\infty}$ algebra, similar analysis shows that the regularized value is $c_{\rm gh}(2) = 0$, [70], as the summation over $j = s - 2$ extends from -1 to infinity, and the critical value turns out to be $c = 0$ in this case.

We now turn to the non-conformal algebra $W_\infty^{(3)}$, which is of interest here, and introduce a pair of ghost fields (b_s, c_s) for each generator with $s \geq 3$. Using the general formulation of this algebra in terms of the structure constants $g_{2r}^{ss'}(n, m; \mu)$ with $\mu = 1/2$, we may write the BRST operator in terms of Fourier modes,

$$\mathcal{Q} = \sum_{s\geq 3} \tilde{V}_m^s c_s^m - \frac{1}{2} \sum_{s,s'\geq 3} \sum_{r\geq 0} g_{2r}^{ss'}\left(n, m; \mu = \frac{1}{2}\right) : c_s^{-n} c_{s'}^{-m} b_{s+s'-2r}^{n+m} : , \tag{7.12}$$

where summation over n and m is also implicitly assumed. Reverting to the coordinate representation as more appropriate for the operator product expansions, we write the BRST current of the algebra in the form

$$j(z) = \sum_{s\geq 3} \tilde{V}^s c_s(z) - \sum_{s,s'\geq 3} \sum_{r\geq 0} f_{2r}^{ss'}(\partial_{c_s}, \partial_{c_{s'}}; \mu = 1/2) : c_s c_{s'} b^{s+s'-2r} : (z) , \tag{7.13}$$

where $f_{2r}^{ss'}$ are given by equations (6.18)-(6.20), as before, and ∂_{c_s} denotes the derivative operator $\partial/\partial z$ acting on the ghost field c_s. The summations also range from 3 to ∞, as dictated by the operator content of the algebra $W_\infty^{(3)}$.

The operator product expansion of the corresponding BRST currents $j(z)j(w)$ follows from general considerations and it consists of the series of terms

$$\begin{aligned} j(z)j(w) \, &= (5!\, c_3 + c_{\rm gh}(3)) \frac{: c_3(z) c_3(w) :}{(z-w)^6} + (7!\, c_4 + c_{\rm gh}(4)) \frac{: c_4(z) c_4(w) :}{(z-w)^8} + \\ &+ (9!\, c_5 + c_{\rm gh}(5)) \frac{: c_5(z) c_5(w) :}{(z-w)^{10}} + \mathcal{O}\left((z-w)^{-12}\right) , \end{aligned} \tag{7.14}$$

where normal ordering is used to extract all singular terms in different orders depending on s. Then, taking into account equation (7.2) for the normalization of the central terms c_s and computing the ghost contribution to the higher spin terms $c_{\rm gh}(s)$, one may verify that the numerical coefficients of the singular terms vanish all at once for the same choice of the central charge c. The nilpotency of the BRST operator should be implemented consistently level by level in s and the critical charge should be the same for all spins. This procedure is rather cumbersome to follow in all generality, since the zeta function

$B_4(x) = x^4 - 2x^3 + x^2 - 1/30$, $B_5(x) = x^5 - 5x^4/2 + 5x^3/3 - x/6$, $B_6(x) = x^6 - 3x^5 + 5x^4/2 - x^2/2 + 1/42$, $B_7(x) = x^7 - 7x^6/2 + 7x^5/2 - 7x^3/6 + x/6$, $B_8(x) = x^8 - 4x^7 + 14x^6/3 - 7x^4/3 + 2x^2/3 - 1/30$, $B_9(x) = x^9 - 9x^8/2 + 6x^7 - 21x^5/5 + 2x^3 - 3x/10$, and so on. In general they satisfy the relation $B_n'(x) = nB_{n-1}(x)$ with $B_n(0)$ equal to the Bernoulli numbers.

regularization of $c_{\rm gh}(s)$ are not available in closed form for arbitrary values of s. We will present the result of explicit calculations for $s = 3, 4$ and 5, which yield the same value $c = 2$ for the corresponding critical charge. Higher spin calculations can also be carried out, and some general arguments about the consistency of the regularization scheme to all levels will be presented later.

Starting with $c_{\rm gh}(3)$, we note that its computation involves contributions from double contractions of the terms $\sum_{s\geq 3} : c_s c_3 b_{s+1} : (z) : c_{s+1} c_3 b_s : (w)$ that appear in the operator product expansion $j(z)j(w)$, with derivatives also distributed on the ghost fields according to equation (7.13). After some calculation, they give rise to the infinite sum

$$\begin{aligned} c_{\rm gh}(3) \;=\; -8\sum_{s=3}^{\infty} \phi_2^{3,s+1}\left(\mu=\frac{1}{2}\right) \Big(& 10s(2s-1)(s-1)^2 + 15(2s-1)(s-1)^2 + 6 + \\ & + 10s(s-1)(2s-1) + 30(s-1)^2 + \\ & + 10(s-1)(2s-1) + 30(s-1)\Big)\,, \end{aligned} \qquad (7.15)$$

where

$$\phi_2^{3s}\left(\mu=\frac{1}{2}\right) = 1 - \frac{15}{(2s-1)(2s-3)}\,. \qquad (7.16)$$

Setting $j = s-3$, we reorganize the sum and compute it as follows, using zeta function regularization,

$$\begin{aligned} c_{\rm gh}(3) &= 16\sum_{j\geq 0}\left(-10(j+3)^4 + 45(j+3)^2 - \frac{77}{4} + \frac{45}{8(2j+7)} - \frac{45}{8(2j+5)}\right) \\ &= 16\left(-10\zeta(-4,3) + 45\zeta(-2,3) - \frac{77}{4}\zeta(0,3) - \frac{9}{8}\right) = -128\,. \end{aligned} \qquad (7.17)$$

The constant term $-9/8$ that appears in the second line has remained after the pairwise cancellations between the last two fractional terms shown in the first line. Then, since the coefficient of the singular term $(z-w)^{-6}$ appearing in equation (7.14) is $5!c_3 + c_{\rm gh}(3) = 64c + c_{\rm gh}(3)$, we conclude that $c = 2$ at this level, as advertised before. It is the first nilpotency condition that determines the critical value of the central charge c of $W_\infty^{(3)}$.

Next, we present some technical details and outline the way to handle the computations for arbitrary spin level in order to achieve the desired cancellation of anomalies when $c = 2$. We set up the general framework for evaluating the ghost contribution to the spin s central charge, following earlier work on infinite dimensional algebras of W_∞-type, [70]. We have, in particular,

$$c_{\rm gh}(s) = -\sum_{r=0}^{s-2}\sum_{s'\geq s_0} \frac{\phi_{2r}^{s,s'+1}(\mu)\,\phi_{2s-2r-4}^{s,s+s'-2r-1}(\mu)}{4(2r+1)!(2s-2r-3)![(2s'-2r-1)!]^2} X(s,s',r;\mu) \qquad (7.18)$$

where $s_0 = \max(2\mu+1, 2r-s+3+2\mu)$ and

$$\begin{aligned} X(s,s',r;\mu) \;=\; & \sum_{k=0}^{2r+1}\;\sum_{k'=0}^{2s-2r-3}\;\sum_{l=0}^{2r+1-k}\;\sum_{l'=0}^{2s-2r-3-k'} (2r+1-l+l')!(2r+k'+1)!(2s'-k)!\,\times \\ & \times\frac{(2s-2r-3-l'+l)!(2s-2r-3+k)!(2s+2s'-2r-4-k')!}{k!\,k'!\,l!\,l'!\,(2r+1-k-l)!(2s-2r-3-k'-l')!}\,. \end{aligned} \qquad (7.19)$$

All these terms arise by making appropriate double contractions of the ghost terms that contribute to $c_{\rm gh}(s)$ for any given spin s, and they are valid for all higher spin truncations of $W_{1+\infty}$ characterized by μ. Of course, here, we must set $\mu = 1/2$ and proceed with explicit calculations.

The ghost contribution to the spin 4 central charge is given by the following three infinite sums

$$
\begin{aligned}
c_{\rm gh}(4) \;=\; & -32\sum_{j\geq 1}\left(21\left(j+\frac{5}{2}\right)^6 - 210\left(j+\frac{5}{2}\right)^4 + \frac{4473}{8}\left(j+\frac{5}{2}\right)^2 - \frac{3321}{8} - \right. \\
& \left. - \frac{4725}{128(2j+3)} + \frac{6075}{64(2j+5)^2} + \frac{4725}{128(2j+7)}\right) - \\
& -64\sum_{j\geq 1}\left(35\left(j+\frac{3}{2}\right)^6 - 245\left(j+\frac{3}{2}\right)^4 + \frac{3675}{8}\left(j+\frac{3}{2}\right)^2 - \frac{495}{8} - \right. \\
& \left. - \frac{7155}{128(2j+1)} - \frac{6075}{128(2j+1)^2} + \frac{7155}{128(2j+5)} - \frac{6075}{128(2j+5)^2}\right) - \\
& -32\sum_{j\geq 3}\left(21\left(j+\frac{1}{2}\right)^6 - 210\left(j+\frac{1}{2}\right)^4 + \frac{4473}{8}\left(j+\frac{1}{2}\right)^2 - \frac{3321}{8} - \right. \\
& \left. - \frac{4725}{128(2j+3)} + \frac{6075}{64(2j+5)^2} + \frac{4725}{128(2j+7)}\right). \qquad (7.20)
\end{aligned}
$$

Each sum has to be evaluated independently by shifting the summation index appropriately and making use of the zeta function regularization. After some calculation the result turns out to be

$$
\begin{aligned}
c_{\rm gh}(4) \;=\; & -64\left(21\zeta\left(-6,\frac{7}{2}\right) - 210\zeta\left(-4,\frac{7}{2}\right) + \frac{4473}{8}\zeta\left(-2,\frac{7}{2}\right) - \right. \\
& - \frac{3321}{8}\zeta\left(0,\frac{7}{2}\right) + 35\zeta\left(-6,\frac{5}{2}\right) - 245\zeta\left(-4,\frac{5}{2}\right) + \\
& \left. + \frac{3675}{8}\zeta\left(-2,\frac{5}{2}\right) - \frac{495}{8}\zeta\left(0,\frac{5}{2}\right) - \frac{3177}{64}\right) = -16^2\cdot 12\,. \qquad (7.21)
\end{aligned}
$$

Since the numerical coefficient of the corresponding term in the operator expansion of the BRST currents is $7!c_4 + c_{\rm gh}(4) = 16^2\cdot 6c + c_{\rm gh}(4)$, we find once more that the anomaly cancellation occurs at $c = 2$ for spin 4.

The ghost contribution to the spin 5 anomaly can be evaluated in a similar way following the general expression (7.18), but the individual terms are quite a lot in number. Here, we only present the end result of the calculation using zeta function regularization of the various sums,

$$
\begin{aligned}
c_{\rm gh}(5) \;=\; & 1024\left(-9\zeta(-8,4) + \frac{615}{4}\zeta(-6,4) - \frac{13245}{16}\zeta(-4,4) + \frac{106815}{64}\zeta(-2,4) - \right. \\
& - \frac{260181}{256}\zeta(0,4) - 63\zeta(-8,3) + \frac{3225}{4}\zeta(-6,3) - \frac{399165}{112}\zeta(-4,3) + \\
& \left. + \frac{2609175}{448}\zeta(-2,3) - \frac{412947}{256}\zeta(0,3) - \frac{13005}{256}\right) = -16^3\cdot\frac{288}{7}\,. \qquad (7.22)
\end{aligned}
$$

Then, since the overall coefficient of the corresponding singular term in the operator product expansion (7.14) is $9!c_5 + c_{\rm gh}(5) = 16^3 \cdot 144c/7 + c_{\rm gh}(5)$, we find again that the algebra is free of anomalies for $c = 2$, as required for consistency at all spin level.

The computation of the higher ghost central charges becomes extremely more complicated as the spin level increases, but in all cases we have examined so far the result is the same and singles out $c = 2$ as the critical value of the non-conformal algebra $W_\infty^{(3)}$. Actually, this is not surprising because the ghost currents satisfy the same operator algebra, and therefore all $c_{\rm gh}(s)$ are also related to each other in terms of a single central charge c, as in equation (7.2) above. Thus, if there is cancellation at a given spin level, it will also hold at all levels. What is rather surprising, however, is the consistency of the regularization scheme based on zeta functions, which seems to respect the structure of the symmetry algebra. It is not guaranteed to be so because there is an intrinsic ambiguity to regularize the divergent sums $\sum_{n\geq 0}(n+a)^{-s}$ when $s = 0$; in particular, $\sum_{n\geq 0} 1$ can be chosen to be equal to $\zeta(0, a) = -B_1(a) = -a + 1/2$, while a remains arbitrary. This arbitrariness may lead to any value of the critical central charge at a given spin level, which then can create discrepancies by comparing the results at different levels or require fine tuning level-by-level.

It is remarkable that there is a natural regularization scheme which appears to be consistent with the nilpotency of the BRST charge and it resolves the ambiguous choice of the free parameter a in all terms that involve $\sum_{n\geq 0} 1$. The prescription we followed simply sets a equal to the value of its companion terms within a given bracket of divergent sums. For instance, $a = 1$ in the bracket of divergent sums (7.9), whereas a is chosen to be 3 in the bracket of divergent sums that appear in (7.17); likewise, a is taken to be 7/2, 5/2 and 7/2 in the three different brackets of divergent sums shown in equation (7.20), respectively. This method works consistently for all W_∞-type algebras, but it has no rigorous foundation to this day. It resembles the description of quantum corrections in Kaluza-Klein theories, where a vanishing one-loop vacuum energy in higher dimensions has a lower dimensional interpretation as a divergent sum that also equals to zero by zeta function regularization. Unfortunately, we do not know of a higher dimensional theory that collects the infinite collection of all two-dimensional higher spin currents into a single entity. If this is properly understood, our results will be put into a firm and better frame. In any case, such higher dimensional interpretation could also be used to provide a more fundamental definition of the tensionless limit.

Finally, we return to the tensionless coset model $\mathrm{SL}(2,\mathbb{R})_2\,/\,\mathrm{U}(1)$ that exhibits the $W_\infty^{(3)}$ symmetry with $c = 1$. As such, it is not free of anomalies but it provides a building block for constructing $c = 2$ models with nilpotent BRST charge. If higher spin symmetries play a fundamental role in the ultimate definition of tensionless strings, the BRST analysis above will serve as a guide for model building. In that case it will be possible to gauge the entire world-sheet symmetry, as it is anomalous free for $c = 2$, and examine its breaking pattern by including $1/\alpha'$ corrections.

8. Generalizations to higher dimensional cosets

Generalizations to higher dimensional coset models can be described in a similar fashion by gauging different subgroups H of higher rank non-compact groups G. Here, we will focus attention on the cosets $\mathrm{SO}(n,1)_k/H_k$ by gauging the maximal compact subgroup $H = \mathrm{SO}(n)$ in a vectorial way so that the signature of the classical geometry is always euclidean; instead, if we consider $H = \mathrm{SO}(n-1,1)$ the construction is similar but the signature will be lorentzian. Note that the critical level is $k = 2$ when $n = 2$ or 3, whereas for $n \geq 4$ the critical level turns out to be $k = n - 1$.

Let us begin with the simplest non-abelian coset model for $n = 3$. Summarizing the results of earlier work, [37], we present the solution to the perturbative beta function equations by including all α' corrections. The exact result is obtained by the hamiltonian method, as in section 2, and leads to

$$ds^2 = 2(k-2)\left(dr^2 + G_{\theta\theta}d\theta^2 + G_{\phi\phi}d\phi^2 + 2G_{\theta\phi}d\theta d\phi\right), \tag{8.1}$$

where the metric components depend on the level as follows,

$$\begin{aligned} G_{\theta\theta} &= \beta^2(r)\left(\tanh^2 r - \frac{1}{k-1}\frac{1}{\cos^2\theta}\right), \\ G_{\phi\phi} &= \beta^2(r)\left(\tanh^2 r\, \cot^2\phi\, \tan^2\theta + \frac{\coth^2 r}{\cos^2\theta} - \frac{1}{k-1}\frac{1}{\cos^2\theta\sin^2\phi}\right), \\ G_{\theta\phi} &= \beta^2(r)\tanh^2 r\, \cot\phi\, \tan\theta\,. \end{aligned} \tag{8.2}$$

The function $\beta(r)$ turns out to be

$$\beta^{-2}(r) = 1 - \frac{1}{k-1}\left(\frac{\coth^2 r}{\cos^2\theta} + \frac{\tanh^2 r}{\sin^2\phi}\left(1 + \cos^2\phi\, \tan^2\theta\right)\right) + \frac{1}{(k-1)^2}\frac{1}{\cos^2\theta\, \sin^2\phi} \tag{8.3}$$

and the dilaton equals to

$$\Phi = \log\left(\frac{\sinh^2 2r\, \sin^2\phi\, \cos^2\theta}{\beta(r)}\right) \tag{8.4}$$

so that $\sqrt{G}\exp\Phi$ is independent of k, as required on general grounds. There is also an anti-symmetric tensor field

$$B_{\phi\theta} = \frac{\beta^2(r)}{2(k-1)}\tanh^2 r\, \cot\phi\, \tan\theta\,, \tag{8.5}$$

which is present for all $k < \infty$.

As $k \to \infty$, $\beta(r) \to 1$ and the classical geometry of the coset $\mathrm{SO}(3,1)_k/\,\mathrm{SO}(3)_k$ simply reads

$$\begin{aligned} ds^2 &= dr^2 + \tanh^2 r\,(d\theta + \cot\phi\tan\theta d\phi)^2 + \frac{\coth^2 r}{\cos^2\theta}d\phi^2\,, \\ \Phi &= \log\left(\sinh^2 2r\sin^2\phi\cos^2\theta\right), \end{aligned} \tag{8.6}$$

whereas the anti-symmetric tensor field vanishes in this case. The signature of target space is clearly $+++$ in the entire region covered by the coordinates (r,θ,ϕ). The geometry also exhibits curvature singularities in places where the string coupling $\exp(-\Phi)$ blows up. On the other hand, extending the validity of the exact quantum geometry to all $k \geq 2$, it is easy to observe that the signature changes in places where $\beta^2(r)$ is not positive definite.

For instance, if r and ϕ are restricted in a region where $\coth^2 r \ \sin^2\phi \geq 1$, whereas θ remains arbitrary, the components $G_{\theta\theta}$ and $G_{\phi\phi}$ will have opposite signs as $k \to 2$. Thus, we encounter a situation which is similar to the vector gauged $\mathrm{SL}(2)_k/\mathrm{U}(1)$ coset model at critical level, as noted in section 2. The range of the coordinate system should be restricted to regions of euclidean signature, or else there is a change of signature due to quantum corrections. Furthermore, the metric is multiplied with $k-2$ and it becomes singular at critical level. This behavior is expected on general grounds and it provides another example of tensionless models.

We may also consider the non-abelian coset model $\mathrm{SO}(3,1)/E(2)$, [48], which is obtained by gauging the non-semisimple subgroup of the Lorentz group in four-dimensional Minkowski space. Recall that the group $\mathrm{SO}(3,1)$ has six generators J_i and K_i with $i = 1, 2, 3$, which correspond to the rigid rotations $\mathrm{SO}(3)$ and the three boosts, respectively. On the other hand, the euclidean group $E(2)$ is generated by the following combinations of Lorentz generators,

$$E_1 = J_1 - K_2\,, \qquad E_2 = J_2 + K_1\,, \qquad E_3 = J_3\,, \tag{8.7}$$

which give rise to a semi-direct product group structure. The generators of translations R^2 in $E(2)$ are null in the four-dimensional fundamental representation of the Lorentz group, since

$$\mathrm{Tr}\, E_1^2 = \mathrm{Tr}\, E_2^2 = E_1^3 = E_2^3 = 0\,, \tag{8.8}$$

whereas E_3 generates the subgroup of rotations $\mathrm{SO}(2)$. Then, the vector gauging of the $\mathrm{SO}(3,1)_k/E(2)$ WZW model can be carried out as usual using gauge connections A, $\bar{A}$ with values in $E(2)$. Parametrizing the elements $g \in \mathrm{SO}(3,1) \simeq \mathrm{SL}(2,C)$ as 4×4 matrices and making appropriate gauge choices, one may eliminate the gauge fields and derive the effective space-time action of the coset model. It turns out that the result is described by the theory of a single free boson with background charge, as in equation (4.5), thus showing that a drastic reduction of dimensions takes place in target space, as in section 4. Therefore, null gauging describes the Liouville field that will decouple at critical level when k is shifted to $k-2$ upon quantization, as before.

Furthermore, the null gauging can be equivalently described by boosting the $\mathrm{SO}(3)$ subgroup of $\mathrm{SO}(3,1)$ and then take the infinite boost limit. More precisely, the boosted version of the non-abelian WZW coset $\mathrm{SO}(3,1)/\mathrm{SO}(3)$ is defined by introducing new generators

$$J_1(\beta) = e^{-\beta K_3} J_1 e^{\beta K_3}\,, \qquad J_2(\beta) = e^{-\beta K_3} J_2 e^{\beta K_3}\,, \qquad J_3(\beta) = J_3\,, \tag{8.9}$$

which amount to considering $\mathrm{SO}(3)_\beta$ with boost parameter β. Of course, there is an isomorphism $\mathrm{SO}(3)_\beta \simeq \mathrm{SO}(3)$ for all values of the deformation parameter β with the exception of the limiting value $\beta \to \infty$, in which case a contraction takes place to $E(2)$. The effective action of the vector gauged WZW model $\mathrm{SO}(3,1)/\mathrm{SO}(3)_\beta$ can be computed in powers of $\exp(-2\beta)$ in the large level and large boost limit. Using appropriate redefinition of variables, as those shown in reference [48], the result is the Liouville action (4.5), plus subleading terms of order $\mathcal{O}(\exp(-2\beta))$, which contain two more additional fields that interact exponentially.

The transformation (8.9) can be extended to the three remaining generators by defining

$$K_i(\beta) = e^{-\beta K_3} K_i e^{\beta K_3} \, . \tag{8.10}$$

In the limit $\beta \to \infty$, there is a contraction of SO(3,1) using the boost which is perpendicular to the spatial direction associated to K_3. Clearly, $K_3(\beta) = K_3$ remains inert and it corresponds to the radial coordinate that parametrizes the Liouville action. Thus, the infinite boost limit selects the non-compact spatial direction r which decouples from the rest, as in the simplest $\mathrm{SL}(2,\mathbb{R})$ case discussed in section 4. The null gauging of WZW models was also considered in reference [71] but in a different way; in that case, the resulting background exhibits a more general Toda-like structure and there is no dimensional reduction to only one Liouville field.

The discussion can be repeated for all $\mathrm{SO}(n,1)_k/\mathrm{SO}(n)_k$ cosets with higher values of $n \geq 4$. The exact form of the metric, dilaton and anti-symmetric tensor fields can be worked out in close form, case by case, following [37]. The resulting expressions are quite lengthy and they are not included here, but it turns out that they all yield tensionless models in the limit $k \to n-1$. The decoupling of gravity manifests in the form of a Liouville field, which is also described by gauging the euclidean group $E(n-1)$; equivalently, it follows by contracting the maximal compact subgroup of $\mathrm{SO}(n,1)$ to $E(n-1)$ in the infinite boost limit. This seems to be a universal result for all maximally gauged WZW models based on $\mathrm{SO}(n,1)_k$ current algebras, [48]. In all cases, one is left with a non-geometric theory of $n-1$ bosons after the decoupling of gravity at critical level. Since the effective description of these remnants is not known, one hopes to gain insight into their quantum theory by appealing to world-sheet methods, as for the simplest $\mathrm{SO}(2,1)/\mathrm{SO}(2) \simeq \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ coset.

It is rather unfortunate that the theory of non-abelian parafermions is still poorly understood beyond the semiclassical limit $k \to \infty$, [30], and there are no exact formulae available for them in the quantum regime. However, it is conceivable that the exact representation of the basic parafermion currents $\Psi^{\alpha}(z) \in \mathrm{SO}(n,1)/\mathrm{SO}(n)$ can be obtained by other means when the level assumes its critical value, depending on n. Experience with the $n = 2$ case suggests that the parafermions should be expressed in terms of derivatives of (multi-component) fermions at critical level, and their operator product expansion should yield the corresponding W-generators as fermion bilinears. These fermions could also be expressed as vertex operators of $n-1$ real scalars, via bosonization, with the introduction of appropriate two-cocycle (twist) factors. Furthermore, the parafermions could be dressed by $\mathrm{SO}(n)$ currents to provide a free field realization of the $\mathrm{SO}(n,1)_k$ current algebra at critical level. This is reminiscent of the fermionic realization of current algebras $\hat{G}_k$ at special values of k, using fermions in various representations of G (see, for instance, [23] and references therein); for example, taking fermions in the adjoint of a compact group G yields representations of $\hat{G}_k$ with $k = g^{\vee}$ as fermion bilinears, whereas for orthogonal groups representations of level 1 are constructed choosing fermions in the fundamental representation. It should be emphasized, however, that the parafermionic representation is different but it has not been spelled out in all generality for all k — in particular for $k = g^{\vee}$, which is of interest here. If this issue is resolved, we will be able to provide explicit construction of the

W-algebra that underlies all non-abelian coset models at critical level. It might turn out that the relevant world-sheet symmetries are matrix generalizations of W_∞, and its higher spin truncations, which have been studied elsewhere in different context, [61, 72].

Finally, it will be interesting to consider more general coset models G/H, where H is not necessarily restricted to the maximal compact subgroup of G, and extend the analysis of their quantum properties at critical level.

9. Conclusions and discussion

We studied a certain class of tensionless models by considering gauged WZW theories for non-compact groups G_k at critical level, $k = g^\vee$, equal to the dual Coxeter number of G. The WZW models are unitary exact conformal field theories for all values of $k > g^\vee$, and $k - g^\vee$ is naturally identified with their tension parameter. The behavior of these theories is somewhat singular at critical values of k because the central charge of the Virasoro algebra becomes infinite, and it is appropriate to rescale the Virasoro generators in order to make it finite. Then, in this limit, the Virasoro algebra contracts to an abelian structure and conformal invariance is lost, thus leading to decoupling of gravity from the spectrum. Apart from this incident, the resulting gauged WZW models make perfect sense as two-dimensional quantum field theories, and they exhibit infinite dimensional world-sheet symmetries generated by higher spin fields. For $\mathrm{SL}(2,\mathbb{R})_2/\mathrm{U}(1)$ the complete structure of the underlying W-algebra was determined and found to correspond to a higher spin truncation of W_∞ by excluding the Virasoro generators.

Gauged WZW models also provide a testing ground for comparing different tensionless limits that often appear in the literature. A natural question that arises in this context is whether the tensionless limit of the quantum theory is equivalent to the quantization of the classical tensionless strings on a given background. Spaces that exhibit no α' corrections, such as flat space or pp-waves, provide exact solutions to the beta-function equations to all orders in perturbation theory, and they are expected to yield the same tensionless theory in either case. However, for spaces that the α' corrections are substantial, as for the gauged WZW models, the two methods need not be equivalent. The simplest example of this kind is the coset $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ at $k = 2$, which defines the quantum tensionless limit of the two-dimensional black-hole model as singular conformal field theory. In this case, the decoupling of gravity manifests as decoupling of the Liouville field, and the operators of the coset model, which include the basic parafermion currents, are faithfully realized without it, in terms of the remaining boson. The same phenomenon can be seen directly at the level of the $\mathrm{SL}(2,\mathbb{R})_k$ current algebra, which is realized in terms of two bosons at $k = 2$, rather than using three bosons as for all other values of $k > 2$. Thus, there is no real remnant of the target space geometry at $k = 2$, although the space looks formally like an infinitely curved hyperboloid, which is obtained by including all α' corrections in the perturbative beta function equations and then letting $k - 2 \sim 1/\alpha' \to 0$. In the same context, the null gauged WZW model $\mathrm{SL}(2,\mathbb{R})_k/E(1)$ yields a Liouville field with infinite background charge at $k = 2$, which describes the gravitational sector that decouples in the tensionless limit.

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The results of this study may be of more general value and provide a way to resolve singularities that appear in the classical theory of gravitation. In fact, the classical gravitational field becomes very strong close to space-time singularities, where string propagation behaves as tensionless theory. These singularities may then be resolved within the complete quantum theory of strings, when appropriately defined by including the effect of higher spin massive states, by simply demanding the decoupling of gravity from all other string states. This novel possibility seems to arise quite naturally in the quantum tensionless limit of WZW models, and it is very different in nature from any other classical considerations of the problem. Thus, it appears that quantum strings in strong gravitational fields will experience no gravity after all. The result may be taken as indication that the tensionless limit of quantum conformal field theories has a topological character. However, it is not clear at this moment how to utilize the infinitely many generators of the world-sheet symmetries of the gauged WZW models in order to give a more precise topological characterization of the resulting two-dimensional, but non-conformal, quantum theories. There might be similarities of these models with the theory of topological orbifolds, which were introduced in the past in an attempt to gain intuition about the unbroken phase of string theory in flat space when $\alpha' \to \infty$, [13]. Work is in progress towards this direction.

Apart from these general issues, there are a few open questions that could be addressed directly in the simplest case of WZW models based on the $\mathrm{SL}(2,\mathbb{R})_k$ current algebra. There is an enhancement of symmetries that follows from the behavior of the Kac-Kazhdan determinant formula of the $\mathrm{SL}(2,\mathbb{R})_k$ current algebra. The determinant vanishes at $k = 2$ and it implies the existence of several null states in the Verma module of the current algebra, [43, 44]; the same conclusion also applies to the field theory of the coset model. It will be interesting to use them systematically in order to provide a complete description of the corresponding WZW models at critical level and clarify their interpretation. Also, one may take advantage of the non-linear algebra $\hat{W}_\infty(k)$ that exists for all values of k in the coset model and extrapolates between the two linear algebras $W_\infty^{(3)}$ and W_∞ at $k = 2$ and ∞, respectively, [55]. This algebra could be used as guide to understand the $1/\alpha'$ corrections away from the critical value of k using world-sheet methods. It may also offer a concrete framework to understand the lifting of degeneracies away from $k = 2$ and explain the mechanism that introduces the coupling to gravity. Needless to say, this is a rather important aspect of our working framework, which should be investigated separately in the future in great detail.

Generalizations to higher dimensional coset models G/H based on higher rank non-compact groups G are also interesting to consider, but they are technically more difficult to treat in detail. They share the essential features of the $\mathrm{SL}(2,\mathbb{R})_k/\,\mathrm{U}(1)$ coset at critical level by the decoupling of gravity, and possibly other fields that depend on the choice of H, as their Virasoro central charge also becomes infinite when $k = g^\vee$. It is rather unfortunate, however, that the quantum theory of non-abelian parafermions is less developed than the abelian case, which prevents us to have explicit expressions in terms of free fields and compute their operator product expansion at critical level. Thus, we only have circumstantial evidence for their behavior in the ultra-quantum limit, and as a result we have no explicit construction of the higher spin W-algebras that arise on the world-sheet in the general case. Actually, it might be possible to obtain an alternative definition of the parafermion currents, which is only valid at critical values of k, without having to go through their exact formulation for arbitrary values of k. In any case, we think that the resulting W-algebras will contain $W_\infty^{(3)}$ among their generators, and possibly many others that depend on the details of the particular coset model. We also hope that the formal BRST analysis of the infinite world-sheet symmetries at critical level, which was performed in detail for the $\mathrm{SL}(2,\mathbb{R})_2/\mathrm{U}(1)$ model, thus rendering $W_\infty^{(3)}$ free of anomalies, can be made more systematic in future formulations of the problem, and it can be generalized to groups of higher rank. The existing results indicate that there is a more general framework at work, which underlies the systematic study of tensionless limit, but it still remains unknown.

There are also other general problems that remain largely unexplored in the present work and deserve special attention in the future. Here, we only present some rough ideas that emerge from the necessity to find a more fundamental definition of the tensionless limit and connect it with some more traditional and better understood structures. The details will be studied separately and appear elsewhere.

First, it will be interesting to apply the representation theory of W_∞-type algebras to the non-conformal symmetries that arise on the world-sheet of the tensionless quantum models. We already know the general theory of quasi-finite representations that consist of highest weight representations with only a finite number of non-zero weights for all higher spin operators, $W_0^s|h\rangle = h_s|h\rangle$, and $W_n^s|h\rangle = 0$ for all $n > 0$, [58, 73, 74]. It is quite interesting that all quasi-finite representations can be obtained by free field realizations, [58, 74], as for the higher spin truncation of W_∞ that arises in gauged WZW models at critical level. As a result, the character formulae are expected to be rather simple. Such representations could also be used to assign higher spin charges to all quantum states that become degenerate in the tensionless limit, and they deserve further study in order to sharpen our current understanding of the whole subject. In this framework, we might also be able to explore the topological character of the quantum theory defined by the coset $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ at critical level, and its generalizations thereof.

Second, there are some intriguing mathematical constructions based on Langlands duality for current algebras that relate small with large α' expansions on the corresponding dual faces, since $(k - g^\vee)^{-1}$ is replaced by $\tilde{k} - \tilde{g}^\vee$ by duality, [75, 76]. This duality, which is essentially a current algebra generalization of the usual electric-magnetic duality for the zero mode (global) algebras, [77], should be properly understood in WZW models in order to reformulate the tensionless limit in more accessible terms. Thus, electric and magnetic charges for loop groups, which are naturally defined for a certain class of periodic instanton configurations in four dimensions, [78], might turn out to play a very important role in future constructions towards a dual formulation of the problem. It should be emphasized, however, that we are only interested in the case of non-compact groups, as the compact models exhibit no tensionless limit. In any case, we expect that these methods can be applied directly to gauged WZW models and lead to a new non-perturbative formulation, where the tensionless limit can be studied more systematically in all generality. Also, in this context, it will be interesting to understand the relation between the world-sheet symmetries of the dual models, when appropriately defined by Langlands duality, and

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generalize earlier mathematical work on the subject, [75, 76]. The results should be able to explain why the black-hole coset $\mathrm{SL}(2,\mathbb{R})_k / \mathrm{U}(1)$ exhibits a symmetry of W_∞ type in both limits, $k \to 2$ and $k \to \infty$, modulo the Virasoro algebra.

Third, the possible connections with non-commutative geometry should be put on a firm basis following the general ideas outlined in reference [19]. In particular, WZW models at large values of the level k are naturally related to the usual commutative geometry in target space, whereas $\alpha' \sim 1/k$ corrections may be viewed as inducing quantum deformations that lead to non-commutative structures. For non-compact groups G_k there is a critical value of the level, $k = g^\vee$, which resembles the infinite non-commutativity limit, and therefore it becomes tractable in many respects. In this case the notion of space-time becomes very fuzzy, as points become totally delocalized, and a new formalism is required to make sense of the underlying structures from a more fundamental viewpoint. Experience with other non-commutative field theories might prove useful in this respect, and the structure of the $\mathrm{SL}(2,\mathbb{R})_k / \mathrm{U}(1)$ model at critical level might resemble the quantum Hall effect in the infinite non-commutativity limit, when the strength of the external magnetic field becomes infinite and the hamiltonian vanishes; recall in this case that the Landau levels become degenerate with zero energy, as the energy levels of the system are proportional to the fundamental frequency $\omega \sim 1/B$, which tends to zero. In any case, we expect that non-commutative geometry could be used further in order to provide an intrinsic definition of the tensionless limit of quantum string theory. Thus, the physics of very strong gravitational fields and the meaning of space-time singularities should be revisited in this context.

In conclusion, the tensionless limit of two-dimensional conformal field theories, and their relevance to the ultimate formulation of string theory beyond the effective field theory description, are interesting problems that deserve better attention. The present work indicates that many surprising things emerge along this path, and new ideas are certainly required in order to put forward some of the results in a systematic way. Also, there might be other tensionless models which are defined by different methods, without ever arising as limiting cases of two-dimensional conformal field theories. The universality of this limit and its fundamental definition in a background independent way remain open problems.

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Note added. It was pointed out by the referee (and other colleagues) that one should rather expect an alternative interpretation of the tensionless limit as a gauge theory of higher spins with huge gauge symmetry and that any consistent theory of massless higher spin fields should also involve gravity. This picture emerges at the free level in Minkowski space (see, for instance, [79], and references therein), and it has been further extended to (A)dS backgrounds using the BRST formalism, as in reference [80]. Thus, if such picture persists in all tensionless models it will raise a puzzle that certainly calls for attention in connection with our results.

We do not have a definite answer to offer at this moment but only a few general comments that indicate the differences with other works and the means of investigation. First, it should be noted that the construction of $W_\infty^{(3)}$ proves the existence of higher spin operator algebras in two dimensions without Virasoro generators. This result might be specific to two-dimensional world-sheet symmetries, but this is precisely were most of our work is confined. Of course, we also have a contracted Virasoro algebra, as in flat space, which can be used at will, but it is completely decoupled from the remaining symmetries of our model. Second, we do not have a target space interpretation of the $W_\infty^{(3)}$ symmetry in order to explore directly the connection with the gauge theory of higher spin fields and their space-time field equations, if appropriate, as in references [79, 80].

However, using the BRST formalism for the algebra $W_\infty^{(3)}$ it is possible to construct lagrangians of the form $\mathcal{L} \sim \langle \Phi | \mathcal{Q} | \Phi \rangle$ and address the problem in all generality, but the results can be rather involved. Yet, it may prove instructive to compare the results with the similar construction based on BRST formalism for the contracted Virasoro algebra, as it is usually done in flat space for the higher spin triplets. Such comparison may very well separate the topological from the gauge theory aspects of higher spin fields in the tensionless limit of gauged WZW models and provide us with a definite answer. On the technical side, the role of the many null states that arise in the representation theory of non-compact current algebras at critical level also needs to be understood in this context in order to build tensors of general type. They rest on special properties of the SL$(2, \mathbb{R})$ current algebra at $k = 2$, which are not shared by the oscillators at generic level. A few steps were already taken by other authors, [20], in an attempt to reproduce the analogue of Fronsdal's conditions at critical level, but the results are still inconclusive: a huge gauge symmetry makes its appearance, as in flat space, but the transversality condition cannot be obtained from the contracted Virasoro constraints.

The above issues constitute open problems for future work and indicate that tensionless WZW models have additional features which are not shared by other models. They should be compared with other instances that involve topological modes of higher spin fields, as in two-dimensional string theory.

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